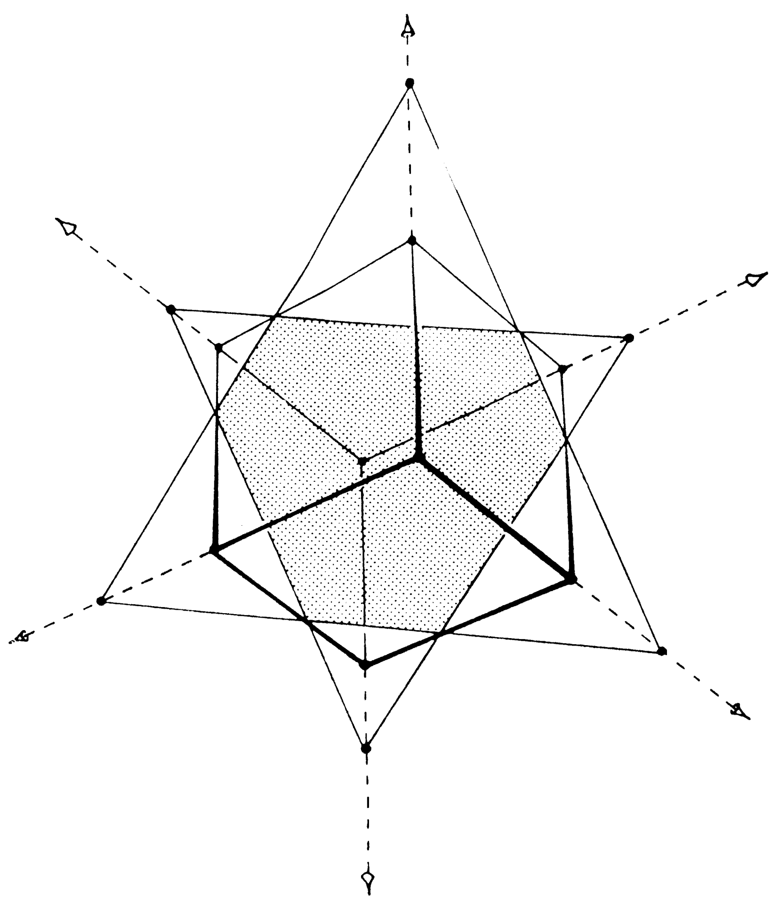
Vol. 64 No. 4, October 1991

# MATHEMATICS MAGAZINE



- Cube Slices, Pictorial Triangles, and Probability
- Napoleon, Escher, and Tessellations
- Power Series Expansions for Trigonometric Functions via Solutions to Initial Value Problems

An Official Publication of The MATHEMATICAL ASSOCIATION OF AMERICA

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## AUTHORS

**Don Chakerian** received his Ph.D. in 1960 from the University of California at Berkeley under the direction of István Fáry (one of whose results plays a role near the end of this article). After three years as an instructor at CalTech he traveled to the University of California at Davis and has been there since, exploring various aspects of convexity. This article results from the longstanding fascination of the two authors with problems fusing geometry and combinatorics.

**Dave Logothetti** received his Ph.D. in 1972 from the University of California at Los Angeles in curriculum and instruction with mathematics as a cognate field, supervised by Louise Tyler, along with Barrett O'Neill and Richard Arens, among others. With Don Chakerian, a mentor, he shares enthusiasm for corny jokes, cartoons, and especially geometry. He is also a disciple of George Pólya, whose influence may be discerned in his three books on teaching mathematics and problem solving in junior and senior high school (one co-authored with his son Ted), and articles published by the California Mathematics Council, the National Council of Teachers of Mathematics, and of course the MAA.

*Dave Logothetti passed away on 20 July 1991. We will miss his keen sense of humor and his mathematical good taste. He was a wonderful referee and a talented writer.*

—*Ed.*

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# MATHEMATICS MAGAZINE

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# ARTICLES

## Cube Slices, Pictorial Triangles, and Probability

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### 1. Introduction

If we slice a cube by planes perpendicular to a body diagonal and passing through vertices, we see in four successive positions cross sections consisting of a point, a "rightside-up" triangle, an "upside-down" triangle, and another point (FIGURE 1). The numerical pattern of the vertices in these cross sections is 1, 3, 3, 1, famous for being a row of Pascal's triangle.

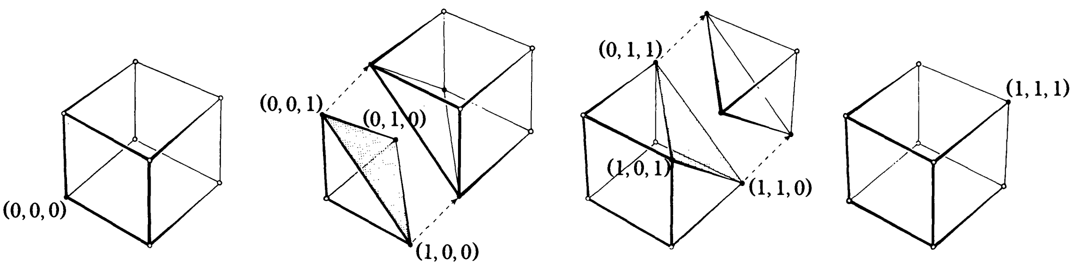


FIGURE 1

Backing down to a 2-dimensional cube, alias a "square," we see three successive cross sections consisting of a point, a line segment, and another point, with the vertices generating the numerical pattern of 1, 2, 1, another row of Pascal's triangle. (See FIGURE 2.) And backing down to 1-dimensional and 0-dimensional cubes ("line segment" and "point"), we analogously find the top two rows of Pascal's triangle.

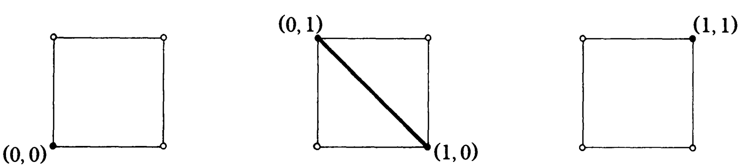


FIGURE 2

We gather these results together in FIGURE 3, showing for $k = 0, 1, 2, 3$, successive slices of the $k$-dimensional cube and the numbers of vertices on the cross sections.

In this article we pursue this pattern into higher dimensions, showing that successive cross sections of an $n$-dimensional cube along a body diagonal and passing through lattice points on hyperplanes perpendicular to this diagonal give rise to a family of figures that can be pictorially generated in a manner analogous to the way the binomial coefficients are generated in Pascal's triangle. In Section 2 we describe how the pictorial analogue of Pascal's triangle in FIGURE 3 continues quite congenially into dimensions $4, 5, 6, \ldots$, with numbers of vertices on successive cross sections reproducing the pattern in the $n$th row of Pascal's triangle (where by convention the top row is the "0-th" row). For example, the successive cross sections perpendicular to a body diagonal of a 4-dimensional cube and containing vertices of the cube are a point, a "rightside-up" tetrahedron, an octahedron, an "upside-down" tetrahedron, and another point, with respective numbers of vertices 1, 4, 6, 4, 1 (as illustrated on the bottom row of FIGURE 5).

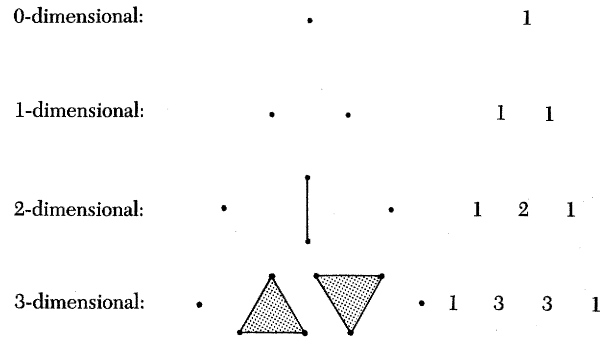


FIGURE 3

Not only do the figures in this pictorial version of Pascal's triangle have numbers of vertices corresponding to the entries in the usual Pascal's triangle, but *each of these figures also can be got by combining the two figures directly above it in a pictorial analogy to Pascal's identity*,

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

In fact, each figure is generated by forming the convex hull of the union of appropriately positioned copies of the two figures in the immediately preceding row. By the "convex hull" of a set S we mean the smallest convex set containing S, i.e., the intersection of all convex sets containing S. The convex hull of a finite collection of points is a closed and bounded convex polyhedron, which we shall call a "polytope." (See Lay **[12]** or Grünbaum **[7]** for a detailed discussion of these matters.) These cross sections of the $n$-dimensional cube constitute a family of polytopes, all but two being $(n-1)$-dimensional (with the first and last 0-dimensional). We denote these cross-sectional polytopes $P(n, k)$, $k = 0, 1, 2, \ldots$, so each $P(n, k)$ has

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

vertices. Thus $P(4, 2)$ has six vertices and turns out, satisfyingly enough, to be a regular octahedron. We discuss the structure of $P(n, k)$ in Section 3. This recursive procedure for drawing the pictures of the $P(n, k)$ (and other clusters of lattice points in later sections) is reminiscent of Pólya's "picture writing" (Pólya **[15]**).

In Section 4 we deal with an $n$-dimensional cube of edgelength 2, subdivided into unit cubes. Successive slices by planes perpendicular to a body diagonal and passing through subdivision vertices give rise to a numerical triangle of special trinomial coefficients, a companion of Pascal's triangle (Pólya **[16**, p. 87**]**) that has in turn a pictorial triangle of its own, illustrated in FIGURE 10.

The vertices of our cubes lie on "lattice points" in $n$-dimensional space: just those points with integer coefficients. In Section 5 we count the lattice points of an $m$-by-$m$-by- $\cdots$ -by-$m$ cube in $n$-dimensional space that lie on one of our slicing planes and obtain further numerical and pictorial triangles of combinatorial interest. It turns out that these numbers of lattice points appear as coefficients of the expansion of $(1+t+t^2+\cdots+t^m)^n$ into powers of $t$. In particular, for $m=1$ we get the usual binomial coefficients in the expansion of $(1+t)^n$, while for $m=2$ we get the trinomial coefficients in the expansion of $(1+t+t^2)^n$, which we explore in Section 4. See FIGURES 12 and 13 for companion numerical and pictorial triangles corresponding to $m=3$.

By counting lattice points on a slice, multiplying by an appropriate factor, and taking a limit, we derive in Section 6 a formula for the $(n-1)$-dimensional volumes (to which we informally refer as "areas") of cube slices perpendicular to a body diagonal. This formula has a longish history and familiar names associated with it, including those of Laplace and Pólya. We apply the formula to the especially interesting central slices connected with a variety of geometric problems, several of which we discuss in Section 11.

In Section 7 we integrate slice areas to get volumes, particularly volumes of slabs of a cube, and these generate further numerical triangles, with Eulerian numbers and their close relatives, Slepian numbers, suddenly appearing on the scene.

We use a volume formula in Section 8 to solve a Putnam problem, and we show in Sections 9 and 10 how these area and volume calculations provide solutions to a pair of appealing problems in geometric probability. For those who wish to investigate geometric interpretations of combinatorial arrays, we recommend the article of Putz [17], wherein geometry and combinatorics are connected differently.

## 2. Cube slices and the pictorial triangle

In order to transcend the inadequacy of our 3-dimensional vision and obtain a clearer view of cross sections of cubes with dimension greater than three, we introduce a coordinate system and don the spectacles of analytic geometry.

Consider the cube in standard position in $n$-dimensional Euclidean space, with vertices at the $2^n$ points with coordinates $(x_1, x_2, \ldots, x_n)$, where each $x_i$ is 0 or 1, to which we will refer as the "unit cube." We want to slice this cube with hyperplanes perpendicular to the diagonal joining $(0,0,\ldots,0)$ to $(1,1,\ldots,1)$. Each such hyperplane has an equation $x_1+x_2+\cdots+x_n=t$, with $t$ varying from 0 to $n$ as we move from the hyperplane through $(0,0,\ldots,0)$ to the parallel hyperplane through $(1,1,\ldots,1)$. Thus, each hyperplane is a translate of the $(n-1)$-dimensional subspace perpendicular to the body diagonal. We shall informally refer to these hyperplanes as "planes" from now on.

These coordinates shed an explanatory light on why Pascal's triangle popped out at us. The plane $x_1+x_2+\cdots+x_n=0$ contains only the vertex $(0,0,\ldots,0)$, while the plane $x_1+x_2+\cdots+x_n=1$, for example, contains the $n$ vertices (lattice points) of the cube having exactly one coordinate 1 and the others 0. In general, the plane $x_1+x_2+\cdots+x_n=k$, $k=0,1,\ldots,n$, contains those vertices $(x_1, x_2, \ldots, x_n)$ having exactly $k$ coordinates equal to 1 and the rest 0. Since each such vertex is determined by choosing $k$ positions for the 1's from the $n$ coordinate slots available, the number of these vertices is $\binom{n}{k}$, read "$n$ choose $k$." So if we denote the $k$th cross section by $\mathrm{P}(n,k)$, $1 \le k \le n-1$, then $\mathrm{P}(n,k)$ is an $(n-1)$-dimensional polytope with $\binom{n}{k}$ vertices, while $\mathrm{P}(n,0)$ and $\mathrm{P}(n,n)$ are the points $(0,0,\ldots,0)$ and $(1,1,\ldots,1)$, respectively.

FIGURE 1 illustrates the case for $n=3$. Reverting to the usual notation in Euclidean 3-space, we see the four successive slices of the unit 3-cube by the planes with equations $x+y+z=k$, for $k=0,1,2,3$. So, for instance, P(3, 2) is the triangle with the $\binom{3}{2}$ vertices (0, 1, 1), (1, 0, 1), and (1, 1, 0) determined by a slice of the plane $x+y+z=2$.

The essence of mathematics is suspicion (with extreme paranoia yielding the best results), and readers should perhaps have a nagging doubt about our assertion that $P(n,k)$ has exactly $\binom{n}{k}$ vertices. We know that the $\binom{n}{k}$ vertices of the cube lying in the plane $x_1+x_2+\cdots+x_n=k$ are indeed vertices of the cross section, but it is not completely obvious that this polytope might not have some other vertices. To see, in fact, that $P(n,k)$ has no vertices that are not already vertices of the $n$-cube, it suffices to note that any vertex of a cross section is precisely where the plane meets a 1-dimensional edge of the cube, so all we need to show is that our plane meets no interior point of an edge. This is easily achieved by mathematical induction; alternatively, we can determine explicitly where a plane meets an edge, which we do in the next section.

The fact that $P(n,k)$ has exactly $\binom{n}{k}$ vertices tells little about its structure in general. We now demonstrate how $P(n+1,k)$ can be constructed from $P(n,k)$ and $P(n,k-1)$ in a natural manner completely analogous to Pascal's identity for generating binomial coefficients,

$$\binom{n+1}{k}=\binom{n}{k-1}+\binom{n}{k}.$$

In order to do this, let us take C to be the unit $(n+1)$-cube (one dimension higher than heretofore) and H to be the plane with equation $x_1+x_2+\cdots+x_{n+1}=k$, $1\le k\le n$. Then $H\cap C=P(n+1,k)$, whose vertices fall into two sets: the first lying in $F_0$, the "bottom" face of C where the final coordinate equals 0, and the second lying in $F_1$, the "top" face of C where the final coordinate equals 1. Observe that $F_0$ is a standard unit $n$-cube, and $H\cap F_0$ consists of those points $(x_1,x_2,\ldots,x_n,0)$ such that $x_1+x_2+\cdots+x_n=k$; i.e., $H\cap F_0$ is a copy of $P(n,k)$. Similarly, $H\cap F_1$ is a copy of $P(n,k-1)$. Those vertices of $P(n+1,k)$ belonging to $F_0$ are precisely the $\binom{n}{k}$ vertices of $H\cap F_0$, and the vertices of $P(n+1,k)$ belonging to $F_1$ are the $\binom{n}{k-1}$ vertices of $H\cap F_1$. FIGURE 4 is a schematic picture of what we have here, with $n=2$ and $k=2$.

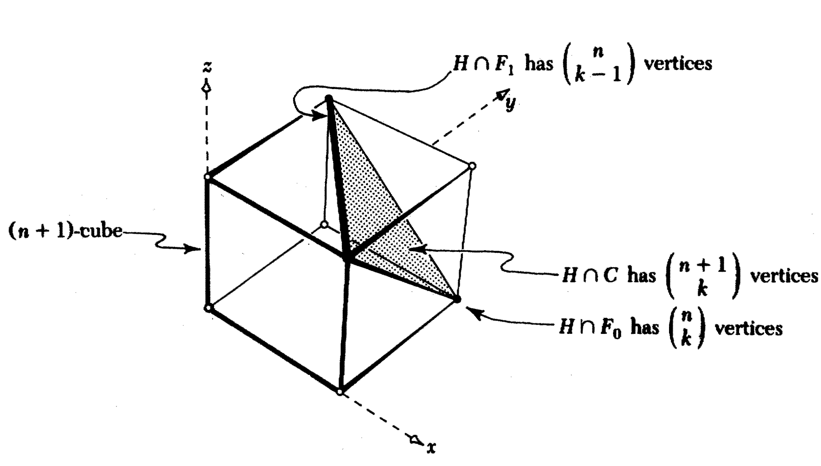


FIGURE 4

Since $P(n+1,k)$ is the convex hull of the set of its vertices, it follows that it is the convex hull of the union of $H \cap F_0$ and $H \cap F_1$. Thus, $P(n+1,k)$ is the convex hull of two appropriately positioned copies of $P(n,k-1)$ and $P(n,k)$.

We repeat ourselves, since this is in a nutshell the main idea of this paper: While in Pascal's triangle we have

$$\begin{matrix} \binom{n}{k-1} & & \binom{n}{k} \\ \searrow & & \swarrow \\ & \binom{n+1}{k} & \end{matrix}$$

meaning we obtain $\binom{n+1}{k}$ by taking the sum of $\binom{n}{k-1}$ and $\binom{n}{k}$, in our pictorial triangle we have

$$\begin{matrix} P(n,k-1) & & P(n,k) \\ \searrow & & \swarrow \\ & P(n+1,k) & \end{matrix}$$

meaning we obtain $P(n+1,k)$ by taking the convex hull of appropriately positioned copies of $P(n,k-1)$ and $P(n,k)$. FIGURE 5 portrays the first five rows of our pictorial triangle, adding one more row to that pictured in FIGURE 3. The last row in this figure gives the cross sections of the unit 4-cube (also called a "tesseract") by the planes $x_1+x_2+x_3+x_4=k$, for $k=0,1,2,3,4$. In the center of that row we have $P(4,2)$, which is a regular octahedron, the convex hull of the properly positioned (in parallel planes) equilateral triangles $P(3,1)$ and $P(3,2)$ from the preceding row. Thus, *the central section of a 4-dimensional cube is a 3-dimensional regular octahedron.*

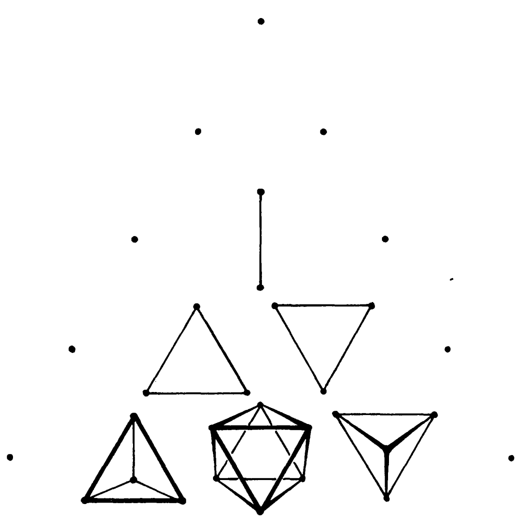

FIGURE 5

The plane $x_1+x_2+x_3+x_4=2$ passes through the center $(1/2,1/2,1/2,1/2)$ of the unit 4-cube. Therefore, the regular octahedron $P(4,2)$ is a "central slice" of the 4-cube, perpendicular to the main diagonal. Such central slices of $n$-cubes will snag our attention later.

Drawing the next row of the pictorial triangle in FIGURE 5, corresponding to $n=5$, would require pictures of six cross sections of the 5-cube, four of which are 4-dimensional and too scary to sketch, although we shall describe them in the following section. We can see at least that for any $n$ the polytopes $P(n,1)$ and $P(n,n-1)$ are regular $(n-1)$-simplices. A regular $(n-1)$-dimensional simplex is the convex hull of $n$ points whose mutual distances apart are equal, this common distance being the "edgelength" of the simplex. Inquisitive readers may check via coordinates of vertices that $P(n,1)$ and $P(n,n-1)$ have edgelength $\sqrt{2}$, for $n=2,3,4,\ldots$.

## 3. Another look at cube slices

We want to get on with the combinatorial and probabilistic aspects of cube slicing; however, it will be useful to pause for an examination of the structure of the polytopes $P(n,k)$ from a new viewpoint. In later sections we will deal with *general* cross sections of an $n$-cube perpendicular to its main diagonal, not necessarily those containing vertices of the cube. For instance, a central slice of a 3-cube perpendicular to its main diagonal is a regular hexagon containing none of the vertices of the cube. FIGURE 6 shows such a slice of the unit 3-cube by the plane $x+y+z=3/2$.

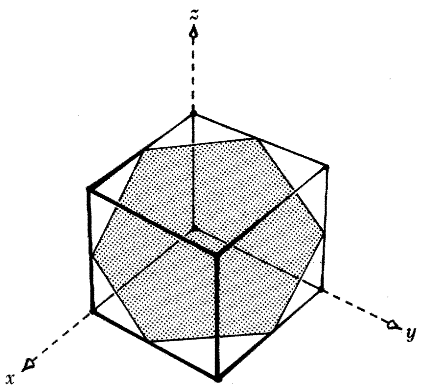


FIGURE 6

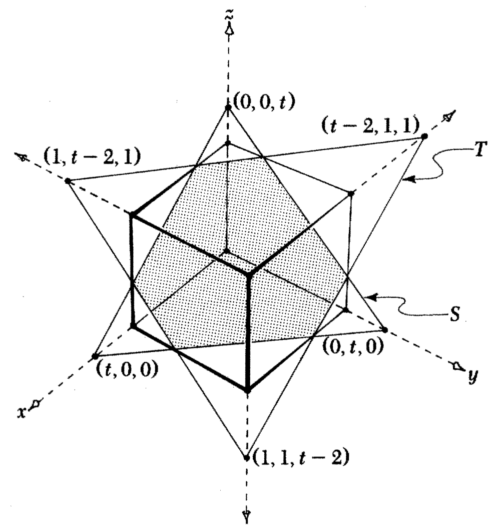


FIGURE 7

It turns out that any slice of an $n$-cube perpendicular to a main diagonal may be viewed as the intersection of two oppositely oriented regular $(n-1)$-dimensional simplices having a common centroid but possibly different edgelengths. The regular hexagon in FIGURE 6 is the intersection of two oppositely oriented equilateral triangles sharing the same centroid. These triangles happen to have the same edgelength since the slice is through the center of the cube.

Equivalently, we can describe each polytope as a truncated regular $(n-1)$-simplex, with the vertices amputated by planes equidistant from and parallel to the opposite faces. This structure may strain our imaginations when amputated portions overlap.

Here we sketch the 3-dimensional case and indicate why the result holds in $n$-dimensions. FIGURE 7 shows a slice of the unit 3-cube C by a plane H with equation $x+y+z=t$, $0<t<3$. Triangle S with vertices $(t,0,0)$, $(0,t,0)$, $(0,0,t)$ lies in H, and the slice $H\cap C$ is just S after its vertices have been amputated by faces of C. So $S\cap C=H\cap C$. Now note that H contains the points $(t-2,1,1)$, $(1,t-2,1)$, $(1,1,t-2)$, and hence triangle T with these points as vertices satisfies $T\cap C=H\cap C$; therefore, slice $H\cap C$ is the same as $S\cap T$. Also observe that triangles S and T have their common centroid (the average of their vertices) at $(t/3,t/3,t/3)$.

All of this can be generalized to $n$-dimensions, with C the unit $n$-cube, H the plane with equation $x_1 + x_2 + \cdots + x_n = t$, $0 \le t \le n$, S the regular $(n-1)$-dimensional simplex with vertices $(t, 0, \ldots, 0), (0, t, \ldots, 0), \ldots, (0, 0, \ldots, t)$, and T the regular simplex with vertices $(t - n + 1, 1, \ldots, 1), (1, t - n + 1, \ldots, 1), \ldots, (1, 1, \ldots, t - n + 1)$. These simplices are oppositely oriented and have their common centroid at $(t/n, t/n, \ldots, t/n)$, with S's edgelength $t\sqrt{2}$ and T's edgelength $(n - t)\sqrt{2}$.

A central slice corresponds to $t = n/2$, in which case both simplices have the same edgelength $n/\sqrt{2}$. For example, the regular octahedron P(4, 2) at the bottom of FIGURE 5 is a central slice of the unit 4-cube. The edgelength of this octahedron is $\sqrt{2}$, and the octahedron is the intersection of two oppositely oriented regular tetrahedra with edgelengths $2\sqrt{2}$ and a common centroid. (See also FIGURE 15.)

The preceding shows that $P(n, k)$ may be viewed as the intersection of a regular $(n-1)$-dimensional simplex of edgelength $k\sqrt{2}$ with an oppositely oriented simplex having the same centroid and edgelength $(n - k)\sqrt{2}$. In case $k = 1$ or $n - 1$, the intersection will be simply the smaller of the two simplices.

Coxeter [4] discusses the polytopes $P(n, k)$, using the following notation for truncations:

$$P(n,k) = \left\{ \begin{matrix} 3^{k-1} \\ 3^{n-k-1} \end{matrix} \right\},$$

and he points out [4, p. 239] how their vertices are distributed among the vertices of a cube.

## 4. The 2-by cube and trinomial coefficients

We now double the edgelength of our cube and cut it into smaller unit cubes. The doubled $n$-dimensional cube we take to have vertices $(x_1, x_2, \ldots, x_n)$, where each $x_i$ is either 0 or 2. This is a "2 by 2 by ... by 2" cube, to which we will refer simply as a "2-by cube." This 2-by cube we cut into $2^n$ unit cubes with $n$ planes parallel to the faces. The vertices of the little cubes are precisely the lattice points contained in the 2-by cube. There are $3^n$ lattice points belonging to the $n$-dimensional 2-by cube, namely those $(x_1, x_2, \ldots, x_n)$ with each $x_i$ equal to 0, 1, or 2. In FIGURE 8 we see the 3-dimensional 2-by cube partitioned into $2^3 = 8$ unit cubes, determining $3^3 = 27$ lattice points.

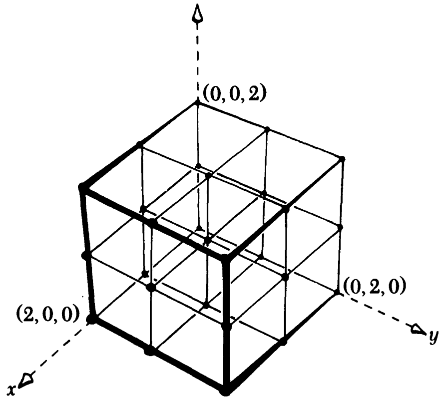


FIGURE 8

We slice the 2-by cube with a plane perpendicular to the main diagonal again, this time counting the number of *lattice points* (rather than *vertices*) in the cube belonging to the plane. Since such a plane laden with lattice points has the equation

$x_1 + x_2 + \cdots + x_n = k$, for some $k = 0, 1, \ldots, 2n$, what we really are doing is counting the number of integral solutions of this equation, with $0 \le x_i \le 2$.

In FIGURE 9 we display the successive slices of the 2-by cube of FIGURE 8 by the planes $x + y + z = k$, $k = 0, 1, \ldots, 6$, showing in each case the lattice points on every slice. The corresponding numbers of lattice points are 1, 3, 6, 7, 6, 3, 1, with a sum of 27.

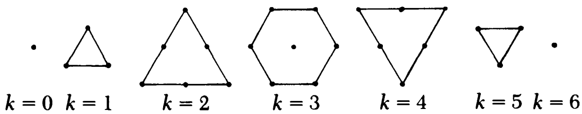


**FIGURE 9**

In FIGURE 10 we show, analogously to FIGURE 3, the 2-by cubes of dimensions 0, 1, 2, 3, the pictorial triangle of cross sections with lattice points, and the corresponding numerical triangle of numbers of lattice points on each slice. (We will shortly show the significance of the dotted "tees"; in the meantime, we trust these dotted tees will not cause crossed eyes.)

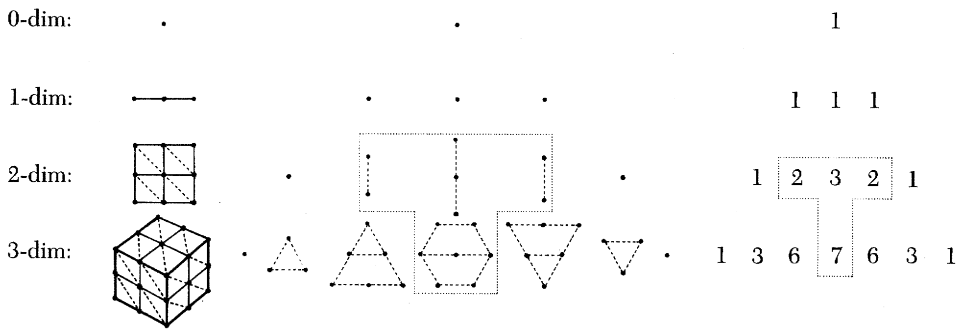


**FIGURE 10**

In FIGURE 10 we have the beginning of a triangle of trinomial coefficients, those that appear in the expansion of $(1 + t + t^2)^n$ into powers of $t$. For example, $(1 + t + t^2)^3 = 1 + 3t + 6t^2 + 7t^3 + 6t^4 + 3t^5 + t^6$, the coefficients of which form the bottom row of the numerical triangle in FIGURE 10, corresponding to dimension $n = 3$.

The generating rule for the numerical triangle of trinomial coefficients resembles the rule for Pascal's triangle, in that each number is the sum of the nearest *three* numbers in the preceding row. This follows directly from the identity, $(1 + t + t^2)(1 + t + t^2)^n = (1 + t + t^2)^{n+1}$. Readers may wish to check that the next row in the trinomial triangle is 1 4 10 16 19 16 10 4 1. Note that in the generating rule a 0 is used for missing numbers at the ends of the preceding row.

As for pictures, the entries of the pictorial triangle in FIGURE 10 are generated analogously, with "convex hull" replacing "sum," just as in the case of the pictorial analogue of Pascal's triangle. For instance, a triangle of lattice points in the last row is obtained by forming the convex hull of appropriately positioned copies of the three figures above it; similarly, the dotted tee shows that the hexagon of 7 lattice points in the middle of the last row is the convex hull of the three clusters above it, corresponding to $7 = 2 + 3 + 2$ in the numerical triangle.

The figure that should appear directly below the hexagon in the pictorial triangle will be one with 19 lattice points, corresponding to the trinomial coefficient in the same place in the numerical triangle. This figure is obtained by slicing the 4-dimensional 2-by cube with the plane $x_1 + x_2 + x_3 + x_4 = 4$. This slice is, again, a regular octahedron, and the resulting cluster of lattice points is generated via the rule for the pictorial triangle as illustrated in FIGURE 11.

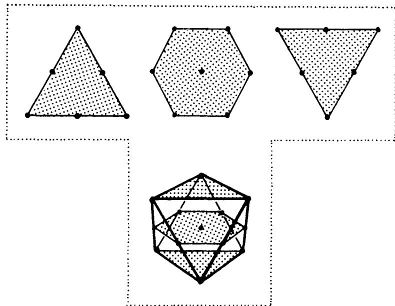

**FIGURE 11**

## 5. Slicing general $m$-by cubes

Let $m$ be a non-negative integer; we shall refer to the $m$ by $m$ by...by $m$ cube with vertices $(x_1, x_2, \ldots, x_n)$ such that each $x_i$ is either 0 or $m$ as the "$n$-dimensional $m$-by cube." This cube is cut into $m^n$ unit cubes by $(m-1)n$ planes parallel to its faces, with the vertices of the little cubes at lattice points that belong to the $m$-by cube. These vertices are the points $(x_1, x_2, \ldots, x_n)$ such that each $x_i$ is one of the numbers $0, 1, \ldots, m$; hence, the total number of such lattice points is $(m+1)^n$.

We now generalize the results of Section 4 by counting the lattice points in the $m$-by cube that lie on the plane $x_1 + x_2 + \cdots + x_n = k$, $0 \le k \le mn$. This number, denoted $N_k(n, m)$, is merely the number of non-negative integral solutions of $x_1 + x_2 + \cdots + x_n = k$, $0 \le x_i \le m$, and is well known to combinatorialists; it is given by the following formula:

$$N_k(n,m) = \sum_j (-1)^j \binom{n}{j} \binom{k+n-1-j(m+1)}{n-1}, \tag{1}$$

where the sum is over $j = 0, 1, \ldots, [k/(m+1)]$, and as usual $[x]$ is the integer part of $x$. This formula's derivation appears in many books on combinatorics (for example, Riordan [18, p. 104] or Vilenkin [22, pp. 98–100]). Since the coefficient of $t^k$ in the expansion of $(1 + t + t^2 + \cdots + t^m)^n$ is precisely the number of ways that $k$ can be represented as an ordered sum of the integers $x_1, x_2, \ldots, x_n$ with $0 \le x_i \le m$, we see that this coefficient is $N_k(n, m)$. Therefore,

$$\frac{(1-t^{m+1})^n}{(1-t)^n} = (1 + t + t^2 + \cdots + t^m)^n = \sum_{k=0}^{\infty} N_k(n,m) t^k. \tag{2}$$

We leave it as reader recreation to use this to derive formula (1). *Hint*: Multiply the binomial expansion of $(1-t^{m+1})^n$ by

$$\frac{1}{(1-t)^n} = \sum_{j=0}^{\infty} \binom{n+j-1}{n-1} t^j. \tag{3}$$

The case $m = 1$ corresponds to slices of the unit $n$-cube, and here we already know that $N_k(n, 1) = \binom{n}{k}$. Thus, we have the *none too obvious* relationship

$$\binom{n}{k} = \sum_{j=0}^{[k/2]} (-1)^j \binom{n}{j} \binom{k+n-2j-1}{n-1}. \tag{4}$$

Let us find the number of lattice points in the 4-dimensional 3-by cube that lie on the plane $x_1 + x_2 + x_3 + x_4 = 5$. This corresponds to $n = 4$, $m = 3$, and $k = 5$, so the number we seek is

$$N_5(4,3) = \sum_{j=0}^{1} (-1)^j \binom{4}{j} \binom{8-4j}{3} = \binom{8}{3} - \binom{4}{1}\binom{4}{3} = 56 - 16 = 40.$$

We can also track down this number in the analogue of Pascal's triangle corresponding to $(1 + t + t^2 + t^3)^n$ shown in FIGURE 12. Note that here to get each entry in the numerical triangle we take the sum of the *four* nearest numbers above it.

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n = 0$: | 1 | | | | | | | | | | | | |
| $n = 1$: | 1 | 1 | 1 | 1 | | | | | | | | | |
| $n = 2$: | 1 | 2 | 3 | 4 | 3 | 2 | 1 | | | | | | |
| $n = 3$: | 1 | 3 | 6 | 10 | 12 | 12 | 10 | 6 | 3 | 1 | | | |
| $n = 4$: | 1 | 4 | 10 | 20 | 31 | 40 | 44 | 40 | 31 | 20 | 10 | 4 | 1 |
| | $k=0$ | $k=1$ | $k=2$ | $k=3$ | $k=4$ | $k=5$ | | | | | | | |

Triangle based on $(1 + t + t^2 + t^3)^n$

FIGURE 12

Let us turn to the geometric analogue of this calculation in terms of slices of the 4-dimensional 3-by cube. FIGURE 13 displays the first four rows of the pictorial triangle of slices of 3-by cubes of dimensions $n = 0, 1, 2, 3$. At the bottom of the figure we have the slice of the 4-dimensional 3-by cube by the plane $x_1 + x_2 + x_3 + x_4 = 5$, showing how it is obtained as the convex hull of appropriately positioned copies of the four nearest figures above it.

Justification of the generating rule for this pictorial triangle is as in §2. The lattice points of the $(n+1)$-dimensional 3-by cube fall into four layers corresponding to points with last coordinate 0, 1, 2, or 3. The slicing hyperplane that yields the figure in position $k$ in the row corresponding to dimension $n+1$ intersects the four layers in the figures that appear in positions $k-3, k-2, k-1$, and $k$ of the row corresponding to dimension $n$.

The pictorial representation extends to all $m$-by cubes in all dimensions $n$. For general $m$ and $n$ the generating rule is given by the recursion relation

$$N_k(n+1, m) = \sum_{j=0}^{m} N_{k-j}(n, m), \tag{5}$$

which can be seen either geometrically as above, or algebraically using the fact that

$$(1+t+t^2+\cdots+t^m)^{n+1}=(1+t+t^2+\cdots+t^m)(1+t+t^2+\cdots+t^m)^n.$$

One last observation about $m$-by cubes: We could have used the generating function

$$(1+w+w^2+w^3)(1+x+x^2+x^3)(1+y+y^2+y^3)(1+z+z^2+z^3),$$

for example, as a labeling device for lattice points in the 4-dimensional 3-by cube, with $w^2xz^2=w^2x^1y^0z^2$, say, corresponding to the point (2, 1, 0, 2). The 40 lattice points belonging to the amputated regular tetrahedron at the bottom of FIGURE 13 correspond to certain monomials of degree 5 in the product. Those on the bottom layer correspond to the last coordinate 0, so the chosen monomials contain only $w$, $x$, $y$, with $z$ to the 0-th power. Similarly, the points in the next layer up have $z$ appearing to the first power, and so on for the other two layers. FIGURE 14 shows the various layers with the $w$, $x$, $y$ parts of the corresponding monomials.

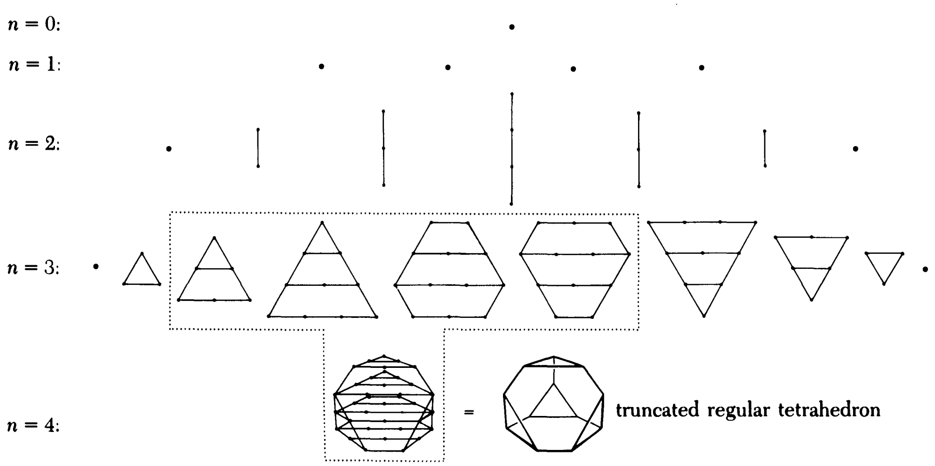


FIGURE 13

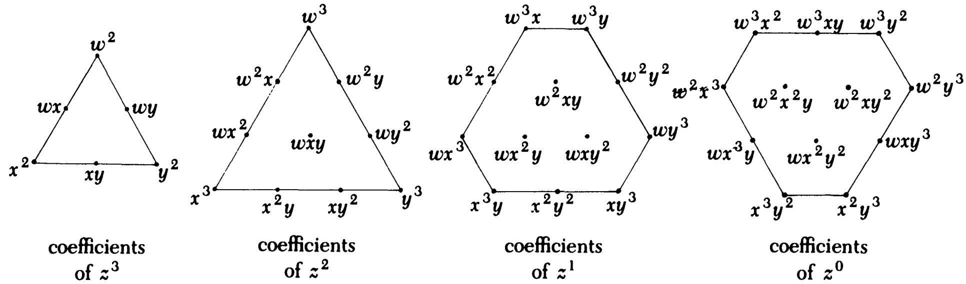


FIGURE 14

## 6. Areas of slices

The results of Section 5 give us a means of calculating the $(n-1)$-dimensional volume (alias "area") of any slice of the unit $n$-cube by a plane perpendicular to the main diagonal. Our method of counting lattice points on a slice and then taking a limit is essentially that used by Pólya [14] in calculating the volume of a truncated $n$-cube. The formulas we derive in this section have a variety of applications to probability (some of which we consider later) and have a history going back to Laplace [10]. Pólya's paper [14] contains references to several authors who have dealt with these questions, including the physicist Arnold Sommerfeld. We shall say a bit more about these matters at the end of this article.

Imagine the unit $n$-cube sliced by the plane with equation $x_1 + x_2 + \cdots + x_n = t$, $0 \le t \le n$; the $(n-1)$-dimensional volume of this slice we shall refer to as the "area" of the slice and denote by A($t$). To get a formula for $A(t)$, we subdivide the cube into $1/m$ by $1/m$ by...by $1/m$ congruent cubes, for $m$ large, and, assuming $t$ is such that the plane contains subdivision points, use the number of lattice points on the hyperplane, multiplied by an appropriate factor, as an approximation for $A(t)$.

Let C be the standard unit $n$-cube, and partition C into $m^n$ congruent little cubes by planes parallel to the faces of C. (We may imagine subdividing a large $m$-by cube, as in Section 5, and then shrinking by a factor of $m$.) The vertices of the little cubes have their coordinates among the numbers $0, 1/m, 2/m, \ldots, (m-1)/m, 1$; we shall refer to these points as "subdivision points."

Now let H be the slicing plane determined by $x_1 + x_2 + \cdots + x_n = t$, with $t = p/q$, where $p$ and $q$ are non-negative integers. If $d$ is the distance from the origin to the point where H intersects the main diagonal of C, and $D$ is the length of this diagonal ($D = \sqrt{n}$), we wish to measure $r = d/D = t/n = p/qn$. We then have that $d = r\sqrt{n} = t/\sqrt{n}$.

To count the number of subdivision points on H, we magnify C by the factor $m$, obtaining the standard $m$-by cube, and count the number of lattice points that belong to the $m$-cube and lie on the image of H under magnification, namely on the plane $x_1 + x_2 + \cdots + x_n = mt$. So that we can use formula (1), which requires $k = mt$ be an integer, we assume $m = Mq$, for some integer $M$. Then $mt = Mqt = Mp$. Thus, the number of lattice points we seek is $N_k(n, m)$, with $k = Mp$ and $m = Mq$, and this is precisely the number of subdivision points on $\mathrm{H} \cap \mathrm{C}$. More conveniently denoting this number by $N$, we have from (1) that

$$N = \sum_{j=0}^{[k/(m+1)]} (-1)^j \binom{n}{j} \binom{k+n-j(m+1)-1}{n-1}. \tag{6}$$

Since $k = mt = rmn$, we have (after some cancellation of factorials)

$$\binom{k+n-j(m+1)-1}{n-1} = \frac{1}{(n-1)!}((rn-j)m - j + (n-1)) \times((rn-j)m-j+(n-2)) \cdots ((rn-j)m-j+(1)). \tag{7}$$

We expand the last product in powers of $m$ and obtain a polynomial of degree $n-1$ in $m$, with leading coefficient $(rn-j)^{n-1}/(n-1)!$. Thus, (6) takes the form

$$N = \frac{m^{n-1}}{(n-1)!} \sum_{j=0}^{[k/(m+1)]} (-1)^j \binom{n}{j} (rn-j)^{n-1} + \cdots, \tag{8}$$

where the later terms are multiplied by powers of $m$ lower than $n-1$.

It can be shown that the plane H is tiled by congruent parallelepipeds having vertices at points $(x_1, x_2, \ldots, x_n)$ with $x_i = 0, 1/m, 2/m, \ldots, (m-1)/m$, or 1, and each parallelepiped having $(n-1)$-dimensional volume $\sqrt{n}/m^{n-1}$. Thus, $N\sqrt{n}/m^{n-1}$

is a good approximation of the area $A(t)$ if $m$ is large. Using (8) we have

$$\frac{N\sqrt{n}}{m^{n-1}} = \frac{\sqrt{n}}{(n-1)!} \sum_{j=0}^{[k/(m+1)]} (-1)^j \binom{n}{j} (rn-j)^{n-1} + \cdots, \tag{9}$$

where the later terms are multiplied by powers of $1/m$ greater than or equal to 1. We now let $M \to \infty$, with $m = Mq$ and $rn = t = p/q$ fixed. The lefthand side tends to $A(t)$, while on the righthand side all terms except the first tend to 0. Since $[k/(m+1)] \to [t]$ as $M \to \infty$, and $k = Mp$ and $m = Mq$, we obtain

$$A(t) = \frac{\sqrt{n}}{(n-1)!} \sum_{j=0}^{[t]} (-1)^j \binom{n}{j} (t-j)^{n-1}, \tag{10}$$

where $t = rn$ is a rational number. Of course, it follows that (10) holds for all $t$ satisfying $0 \le t \le n$. Another way to interpret the sum in (10) is to observe that we add over $j = 0, 1, 2, \ldots,$ stopping when $t - j$ becomes negative.

For those who have taken to heart our injunction that the essence of mathematics is suspicion and who lack faith in our derivation of (10), we do *not* offer an alternative proof, but instead employ a much more powerful method for instilling conviction: We present some *examples*.

Let us begin by calculating the 3-dimensional volume of a central slice of the unit 4-cube. For the slicing plane with $x_1 + x_2 + x_3 + x_4 = t$, $r = t/4$ represents the proportion of the diagonal cut off by the plane. So for a central slice, $r = \frac{1}{2}$, and $t = 2$. Then (10) gives

$$A(2) = \frac{1}{3} \sum_{j=0}^{2} (-1)^j \binom{4}{j} (2-j)^3 = \frac{1}{3}(2^3 - 4 \cdot 1^3) = \frac{4}{3}. \tag{11}$$

Furthermore, recall from Section 3 that a central slice of the unit 4-cube is a regular octahedron of edgelength $\sqrt{2}$; it is the intersection of two oppositely oriented regular tetrahedra of edgelength $2\sqrt{2}$, as indicated in FIGURE 15.

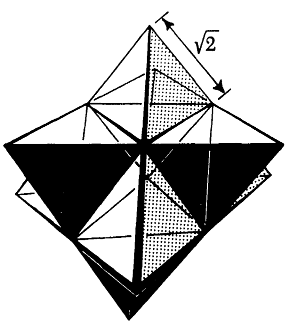


**FIGURE 15**

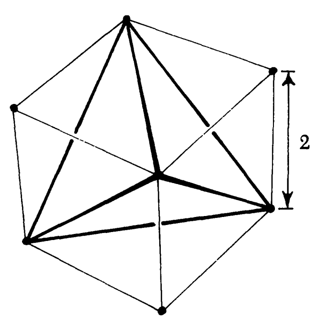


**FIGURE 16**

Inasmuch as the octahedron is the result of amputating from either of the large tetrahedra four little tetrahedra of half the edgelength, the volume of the octahedron is

$$T - 4 \cdot \frac{T}{8} = \frac{T}{2}, \tag{12}$$

where $T$ is the volume of the tetrahedron. But a regular tetrahedron of edgelength $2\sqrt{2}$ sits comfortably in a cube of edgelength 2, shown in FIGURE 16, as a consequence of which we may verify that $T=\frac{8}{3}$, so that from (12) we get $\frac{4}{3}$ for the volume of the octahedron, in confirmation of result (11).

The difference in (12) is identical to the difference in (11) with $T=\frac{8}{3}$. In fact, for $1\le t\le 2$ the sum in (10) takes the form

$$\frac{\sqrt{n}}{(n-1)!}\left(t^{n-1}-n(t-1)^{n-1}\right), \tag{13}$$

representing the volume of an $(n-1)$-dimensional regular simplex of edgelength $t\sqrt{2}$ minus the volumes of $n$ simplices of edgelength $(t-1)\sqrt{2}$, based on the fact that the $(n-1)$-dimensional volume of a regular $(n-1)$-dimensional simplex of edgelength $s$ is [4, p. 295]

$$\sqrt{\frac{n}{2^{n-1}}}\,\frac{s^{n-1}}{(n-1)!}. \tag{14}$$

As we know from Section 3, the slices of the unit $n$-cube are amputated regular simplices. When the cut-off portions overlap, we cannot compute the volume by a single subtraction as in (13), but instead must use the "inclusion-exclusion principle," which is precisely what formula (10) exhibits. It represents the volume of a large simplex, minus the volumes of $n$ amputated simplices, plus the $\binom{n}{2}$ volumes of simplices formed by overlaps corresponding to edges, minus the $\binom{n}{3}$ volumes of simplices formed by overlaps corresponding to 2-dimensional faces, and so forth. This principle can be used to give another proof of (10) and is discussed in Pólya [14] in a slightly different form.

As another application of (10) we calculate the 3-dimensional volume of a slice of the unit 4-cube by a plane perpendicular to the diagonal and $r=\frac{5}{12}$ of the way from the origin to the opposite vertex. This is a slice by the plane with $x_1+x_2+x_3+x_4=t=4r=\frac{5}{3}$. Thus, (10) yields

$$A\left(\frac{5}{3}\right)=\frac{1}{3}\sum_{j=0}^{1}(-1)^j\binom{4}{j}\left(\frac{5}{3}-j\right)^3=\frac{1}{3}\left(\left(\frac{5}{3}\right)^3-4\left(\frac{2}{3}\right)^3\right)=\frac{31}{27}. \tag{15}$$

To reassure ourselves with independent verification, we note that this slice is similar to the amputated regular tetrahedron at the bottom of FIGURE 13 and has one-third of its edgelength. For further reader recreation, check that the required volume is that of a regular tetrahedron of edgelength $5\sqrt{2}/3$ with four regular tetrahedra of edgelength $2\sqrt{2}/3$ cut off from its vertices.

## 7. Volumes of slabs

Certain applications to probability problems require calculation of the volume of a "slab" of a cube, i.e., the volume of the portion between two planes perpendicular to a main diagonal. This can be found by integrating the cross-sectional area over an appropriate range of values; all we need do is calculate the volume of the part of the cube on one side of a plane and then find the volumes of slabs by subtraction.

Let C be our standard unit cube, and H$(t)$ be the plane with $x_1+\cdots+x_n=t$, $0\le t\le n$. Since the distance from the origin to H$(t)$ is $t/\sqrt{n}$, the volume between H$(t)$ and H$(t+\Delta t)$ is approximately

$$\Delta V(t)=\frac{A(t)\,\Delta t}{\sqrt{n}},$$

where $A(t)$ is the cross-sectional area given in (10). Thus, if we let $V(t)$ be the volume of the part of the cube between the origin and H($t$), we get via integration

$$V(t) = \frac{1}{\sqrt{n}} \int_0^t A(u)\, du = \frac{1}{n!} \sum_{j=0}^{[t]} (-1)^j \binom{n}{j} (t-j)^n. \tag{16}$$

This formula, which goes back to Laplace **[10]**, was obtained directly by Pólya **[14]** counting lattice points and taking a limit.

We pause to consider some special cases. If $0 \le t \le 1$, the plane H($t$) cuts off a corner of the cube at the origin. The piece cut off is a non-regular $n$-dimensional simplex with $n$ mutually perpendicular edges of length $t$ emanating from its vertex at the origin (for $n = 3$, just a "trirectangular tetrahedron"). This simplex has $n$-dimensional volume $t^n/n!$, which is exactly what (16) gives for these values of $t$.

When $t = n/2$, the plane cuts the cube in half, so $V(n/2) = \frac{1}{2}$. If we use this in (16) we obtain

$$\sum_{j=0}^{[n/2]} (-1)^j \binom{n}{j} (n-2j)^n = (2^{n-1}) n!, \tag{17}$$

one of the formulas found in Laplace **[10**, p. 171].

The probability problem we consider in the next section requires the volume of a unit cube's slab of a specified width. We need the volume of the standard unit $n$-cube that lies between the planes H($k - \frac{1}{2}$) and H($k + \frac{1}{2}$), where $k$ is a given integer, $0 \le k \le n$. Using (16), we find

$$V(k+\tfrac{1}{2}) - V(k-\tfrac{1}{2}) = \frac{1}{n!} \sum_{j=0}^{k} (-1)^j \binom{n}{j} (k + \tfrac{1}{2} - j)^n$$
$$- \frac{1}{n!} \sum_{j=0}^{k-1} (-1)^j \binom{n}{j} (k - \tfrac{1}{2} - j)^n, \tag{18}$$

which after fiddling with sums, changing index, applying Pascal's identity, and simplifying becomes

$$V(k+\tfrac{1}{2}) - V(k-\tfrac{1}{2}) = \frac{1}{n!\,2^n} \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (2k - 2j + 1)^n. \tag{19}$$

For example, when $n = 3$, we find the volumes of the four slabs of the unit cube between the planes H($-\frac{1}{2}$), H($\frac{1}{2}$), H($\frac{3}{2}$), H($\frac{5}{2}$), H($\frac{7}{2}$) to be

$$\frac{1}{48}, \quad \frac{23}{48}, \quad \frac{23}{48}, \quad \frac{1}{48}. \tag{20}$$

The slabs in question are indicated in FIGURE 17.

Ignoring the factor $1/n!\,2^n$, we find the integers obtained from (19) form an interesting numerical triangle. We denote these numbers $S(k, n-k)$, in honor of David Slepian, who rediscovered these volume formulas and some of the associated combinatorics over three decades ago and wrote them up in an unpublished technical memorandum **[19]**. We have from (19)

$$V(k+\tfrac{1}{2}) - V(k-\tfrac{1}{2}) = \frac{1}{n!\,2^n} S(k, n-k). \tag{21}$$

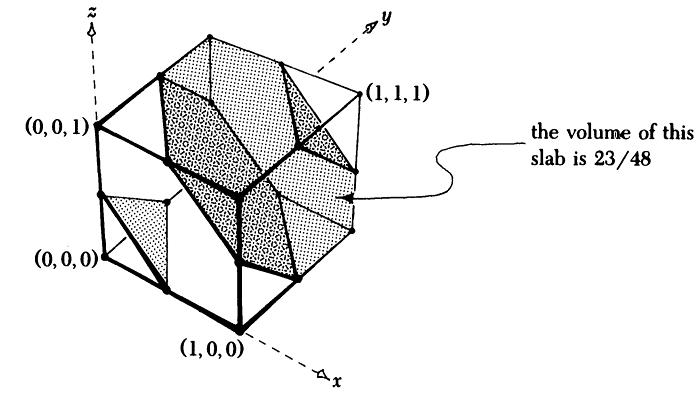


FIGURE 17

Then $S(0,0)=1$; $S(1,0)=S(0,1)=1$; $S(2,0)=S(0,2)=1$, and $S(1,1)=6$, while the numerators in (20) are $S(3,0)$, $S(2,1)$, $S(1,2)$, and $S(0,3)$, respectively. We can arrange these numbers in a triangle as in Figure 18, with each row corresponding to a fixed value of $n$.

The arrows and little numbers in the array indicate the generating rule for the triangle of the $S(k,n-k)$. For example, to get $S(3,1)=S(3,4-3)=76$, we take $3(23)+7(1)$, using the nearest pair in the preceding row. This Pascalish generating rule is expressed as

$$S(k,n-k)=(2n-2k+1)S(k-1,n-k)+(2k+1)S(k,n-k-1). \quad (22)$$

Slepian also considered an associated numerical triangle of numbers $R(k,n-k)$ defined by

$$R(k,n-k)=\sum_{j=0}^{k}(-1)^j\binom{n}{j}(k-j)^{n-1}. \quad (23)$$

In view of (10), $\sqrt{n}\,R(k,n-k)/(n-1)!$ is the area of the slice of the unit $n$-cube by the plane $x_1+\cdots+x_n=k$. We could use (16) to verify that $R(k+1,n-k)/n!$ is also the volume of that part of the unit $n$-cube between the planes with $x_1+\cdots+x_n=k$ and $x_1+\cdots+x_n=k+1$. We have $R(1,1)=1$; $R(1,2)=R(2,1)=1$; $R(3,1)=R(1,3)=1$, and $R(2,2)=4$. Figure 19 shows the numerical triangle of these numbers.

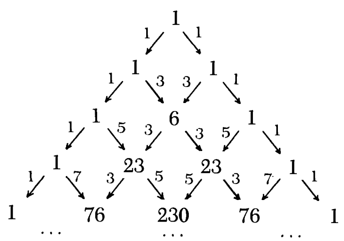


FIGURE 18

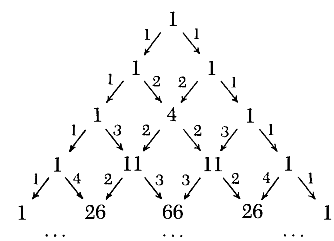


FIGURE 19

Again, the arrows and little numbers give the generating rule for the array. For example, $R(2,4)=26=4(1)+2(11)=4R(1,4)+2R(2,3)$, and the general rule is

$$R(k,n-k)=(n-k)R(k-1,n-k)+kR(k,n-k-1). \quad (24)$$

Note that the $n$th row in FIGURE 19 corresponds to dimension $n$, and $k$ ranges from 1 to $n$.

Experts in combinatorics will recognize (as did Slepian) that the numbers in FIGURE 19 are the "Eulerian numbers." (See Sloane **[20]** or Riordan **[18]**.) The fact that the sum of the entries in the $n$th row is $n!$ corresponds geometrically to the fact that the entries divided by $n!$ give the volumes of successive slabs of the unit $n$-cube, and the sum of these volumes is 1. A modest variety of secrets in the Eulerian triangle is uncovered in Logothetti **[13]**.

Compulsive readers may use (10) to verify Slepian's observation that the central slice of a unit $n$-cube has $(n-1)$-dimensional volume $\frac{\sqrt{n}}{(n-1)!}R(n/2, n/2)$, if $n$ is even, and $\frac{\sqrt{n}}{2^{n-1}(n-1)!}S((n-1)/2,(n-1)/2)$, if $n$ is odd.

## 8. A Putnam problem

The 1976 William Lowell Putnam Competition problem B-5 requests evaluation of $\Sigma_{j=0}^{n}(-1)^j\binom{n}{j}(t-j)^n$. The resemblance of this sum to the sum in (16) suggests we take the high road, a geometric approach.

Since the two planes with equations $x_1 + \cdots + x_n = t$ and $x_1 + \cdots + x_n = n - t$ are equidistant from the center of the unit $n$-cube, we have for the truncated volumes $V(t) + V(n-t) =$ (volume of unit cube) $= 1$. Using formula (16) we get

$$\sum_{j=0}^{[t]}(-1)^j\binom{n}{j}(t-j)^n + \sum_{j=0}^{[n-t]}(-1)^j\binom{n}{j}(n-t-j)^n = n!.$$

Changing the second index from $j$ to $n-j$ gives that second sum the same form as the first, but summed from $n-[n-t]$ to $n$. Worthy readers will then find that the sums combine to give

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}(t-j)^n = n!.$$

Since the expression on the left is a polynomial, and this holds for $0 \le t \le n$, we see that the result holds for all $t$.

## 9. The probability of round off

Hilton and Pedersen **[9]** give the following example of a less conventional school problem with useful arithmetic implications and a striking geometric solution.

> Suppose numbers $x$ and $y$ are selected randomly from the closed interval $[0,1]$. Then the integer nearest to $x+y$ must be one of 0, 1, or 2. What are the respective probabilities of these outcomes?

By looking at FIGURE 20, taken from **[9]**, we immediately see that the probability of $x+y$ rounding off to $k$ is the area of the region labeled $k$.

Therefore, if $P(k)$ is the probability of $x+y$ rounding off to $k$, we have $P(0) = P(2) = 1/8$ and $P(1) = 6/8$. The fact that the numerators reproduce the row corresponding to $n = 2$ in FIGURE 18 is no coincidence, as a glance at equation (21) and some concerted thought show. This generalizes to more than two numbers:

Let $n$ be fixed, and suppose numbers $x_1, \ldots, x_n$ are randomly selected from the closed interval $[0, 1]$. Then the probability that $x_1 + \cdots + x_n$ rounds off to $k$ is $S(k, n-k)/n!\,2^n$.

This follows because those $(x_1, \ldots, x_n)$ that belong to the unit $n$-cube and have $x_1 + \cdots + x_n$ rounding off to $k$ are precisely those points of the cube satisfying $k - \frac{1}{2} \le x_1 + \cdots + x_n \le k + \frac{1}{2}$. The ratio of this slab's volume to that of the entire cube is just $V(k + \frac{1}{2}) - V(k - \frac{1}{2})$, as in (21).

The row of numbers (20) gives the probabilities that $x + y + z$ rounds off to 0, 1, 2, or 3, respectively, when $x$, $y$, $z$ are randomly chosen from $[0, 1]$, and the geometric interpretation is embodied in FIGURE 17.

In the language of probability, the function $V(t)$ in equation (16) is simply the "cumulative distribution function for the sum of $n$ independent random variables uniformly distributed in the interval $[0, 1]$." This explains the success of probabilistic arguments in connection with the geometric problems of Section 11. The full story is told in Feller [**6**, I.9].

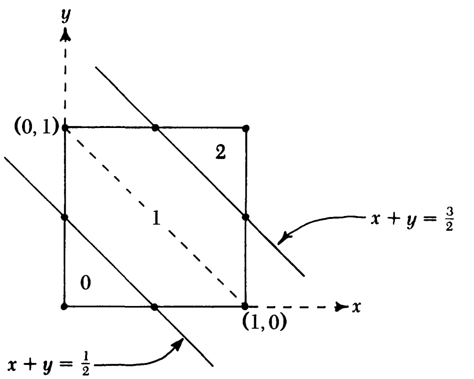


FIGURE 20

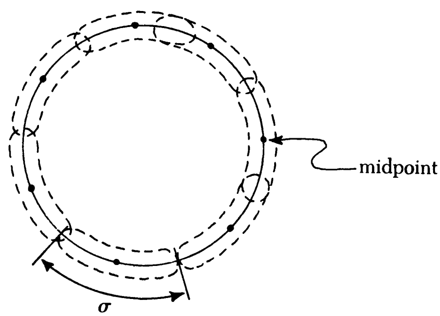


FIGURE 21

## 10. The probability of a (non-governmental) cover-up

The following geometric probability problem and its generalizations are completely treated in Solomon [**21**, Chap. 4], which also tells the interesting history of the problem. Feller [**6**, p. 28] treats the problem in the context of a discussion of the function in (16) as a distribution function.

Let K be a circle of *circumference* 1, and suppose $n \ge 2$ arcs of length $\sigma$ are distributed randomly over K. What is the probability that these arcs cover K? (See FIGURE 21.)

If K is covered, the midpoints of the $n$ arcs will subdivide the circumference of K into $n$ arcs of lengths $x_1, \ldots, x_n$, with $x_1 + \cdots + x_n = 1$. The circle is covered precisely when $x_i \le \sigma$ for all $i = 1, 2, \ldots, n$. Therefore, an equivalent formulation of the problem is the following:

Given $x_1, \ldots, x_n$ satisfying $0 \le x_i \le 1$, $i = 1, 2, \ldots, n$, and $x_1 + \cdots + x_n = 1$, what is the probability that $x_i \le \sigma$ for all $i$?

The set of such $(x_1, \ldots, x_n)$ is the $(n-1)$-dimensional regular simplex S of edgelength $\sqrt{2}$ in $n$-dimensional Euclidean space with vertices at $(1,0,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,0,0,\ldots,1)$. The intersection of S with the standard $n$-cube $C_\sigma$ of edgelength $\sigma$ consists of those $(x_1,\ldots,x_n) \in S$ that satisfy $0 \le x_i \le \sigma$, for $i = 1,2,\ldots,n$. Hence, the desired probability is the ratio of the $(n-1)$-dimensional volumes,

$$P(\sigma) = \frac{A(S \cap C_\sigma)}{A(S)}. \tag{25}$$

To calculate $A(S \cap C_\sigma)$, magnify the figure by the factor $1/\sigma$. We then have the standard unit $n$-cube intersected by the plane with $x_1 + \cdots + x_n = 1/\sigma$. From (10) we know that the magnified intersection has $(n-1)$-dimensional volume

$$\frac{\sqrt{n}}{(n-1)!}\sum_{j=0}^{[1/\sigma]}(-1)^j\binom{n}{j}\left(\frac{1}{\sigma}-j\right)^{n-1} = \frac{\sqrt{n}}{\sigma^{n-1}(n-1)!}\sum_{j=0}^{[1/\sigma]}(-1)^j\binom{n}{j}(1-\sigma j)^{n-1} \tag{26}$$

Since this was obtained after magnification by $1/\sigma$, the original $(n-1)$-dimensional volume, $A(S \cap C_\sigma)$ is found after division by $(1/\sigma)^{n-1}$. But from (14), with $s = \sqrt{2}$,

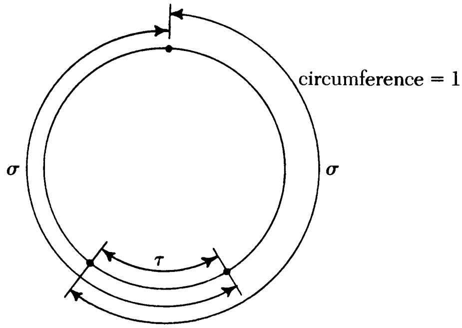


FIGURE 22

we have $A(S) = \sqrt{n}/(n-1)!$. Thus, dividing by these factors in (26), we get the probability we want, *viz.*

$$P(\sigma) = \sum_{j=0}^{[1/\sigma]}(-1)^j\binom{n}{j}(1-\sigma j)^{n-1}. \tag{27}$$

Note that we must have $\sigma > 1/n$ in order that the probability of coverage be nonzero, so it is assumed in (27) that $1/\sigma < n$. Geometrically, this corresponds to the assumption that $S \cap C_\sigma$ is more than a point.

Applying (27) in the case $n = 2$ gives $P(\sigma) = 2\sigma - 1$. This is plausible, since we may see with a sketch that if we fix one arc of length $\sigma$, then the set of centers of other arcs of length $\sigma$ that together with the fixed arc give coverage is itself an arc of length $\tau = 2\sigma - 1$. (See FIGURE 22.)

## 11. Cube slicing, from the nineteenth century to the present

Laplace [**10**, p. 170] gave an integral formula that, in our setup, amounts to

$$A(t) = \frac{2\sqrt{n}}{\pi}\int_0^\infty \left(\frac{\sin u}{u}\right)^n \cos((n-2t)u)\,du, \tag{28}$$

where $A(t)$ is the $(n-1)$-dimensional volume of the slice of the unit $n$-cube as in (10). Pólya [14] proved a general integral formula that gives areas of cube slices by ***arbitrary*** planes, reducing to (28) when the slice is perpendicular to a main diagonal. Our derivation of (10) is similar in spirit to Pólya's, involving counting lattice points and taking a limit.

The distance from the plane with $x_1 + \cdots + x_n = t$ to the origin is $t/\sqrt{n}$. If $t \geq n/2$, then $s = t/\sqrt{n} - \sqrt{n}/2$ is the distance from the plane to the center of the cube. In terms of this distance, (28) becomes

$$A\left(\frac{n}{2} + s\sqrt{n}\right) = \frac{2\sqrt{n}}{\pi} \int_0^\infty \left(\frac{\sin u}{u}\right)^n \cos(2\sqrt{n}\, su)\, du. \tag{29}$$

Laplace and Pólya both gave proofs that

$$\lim_{n \to \infty} A\left(\frac{n}{2} + s\sqrt{n}\right) = \sqrt{\frac{6}{\pi}}\, e^{-6s^2} \tag{30}$$

The case of a central slice is especially interesting. Here $s = 0$, and we obtain

$$\lim_{n \to \infty} A\left(\frac{n}{2}\right) = \sqrt{\frac{6}{\pi}} \doteq 1.382. \tag{31}$$

Thus, our $(n-1)$-dimensional volume of the central slice of the unit $n$-cube (perpendicular to a main diagonal) approaches $\sqrt{6/\pi}$ as $n$ approaches $\infty$. So from (28) we have

$$\lim_{n \to \infty} \frac{2\sqrt{n}}{\pi} \int_0^\infty \left(\frac{\sin u}{u}\right)^n du = \sqrt{\frac{6}{\pi}}. \tag{32}$$

Using probabilistic methods, Hensley [8] showed that there exists an upper bound, independent of $n$, on the $(n-1)$-dimensional volumes of *all* central slices of a unit $n$-cube (not just those perpendicular to a body diagonal). His conjecture that $\sqrt{2}$ is such an upper bound (and this is best possible, since a central slice of an $n$-cube that contains an $(n-2)$-dimensional face has $(n-1)$-dimensional volume $\sqrt{2}$) was proved by Ball [1] in 1986. Both Ball and Hensley used probabilistic methods, ending up making ingenious estimates on integrals corresponding to the integral formula for volume treated by Pólya.

In a later paper, Ball [2] observed that this bound on areas of slices provides a remarkably simple solution, at least for dimension $n \geq 10$, to a famous problem of Busemann and Petty: Must an $n$-dimensional centrally symmetric convex body have greater volume than another such body with the same center if each slice through the center of the first body has greater $(n-1)$-dimensional volume than the corresponding slice through the second? Ball observed that for $n \geq 10$ each central slice of an $n$-dimensional ball of unit volume has $(n-1)$-dimensional volume greater than $\sqrt{2}$, and hence greater than the corresponding slice of the unit cube. Thus, a slightly smaller ball will still have its slices larger than those of the cube, yet have smaller volume. This lays to rest the cases for $n \geq 10$, but the question is still open in dimensions $3, 4, \ldots, 9$. Larman and Rogers [11] earlier gave a more complicated probabilistic argument that settled the problem for $n \geq 12$.

The formula for the volume of an $n$-cube truncated by an arbitrary plane was also treated with brevity and elegance by Barrow and Smith [3] using spline notation. They gave a combinatorial version, similar to a formula of Pólya [14], and indicated the probabilistic significance.

We now come to some discussion of central symmetry that, while a little specialized, connects well with the rest of this paper, is difficult to find in the literature, and gives well-deserved publicity to an interesting unsolved problem.

The central slice perpendicular to the main diagonal of the unit $n$-cube is also of interest in connection with a problem of Fáry and Rédei **[5]**. Recall from Section 3 that such a slice is the intersection of two oppositely oriented regular $(n-1)$-dimensional simplices having the same centroid and edgelength $n/\sqrt{2}$. FIGURE 15 illustrates the case $n=4$. It happens that this intersection (a regular octahedron in FIGURE 15) is the centrally symmetric convex body of largest volume that fits inside the simplex. The ratio of this intersection's volume to that of the simplex tells us, in a sense, how close the simplex is to being centrally symmetric. For example, in FIGURE 23 we see that this "measure of central symmetry" for an equilateral triangle is $\frac{2}{3}$.

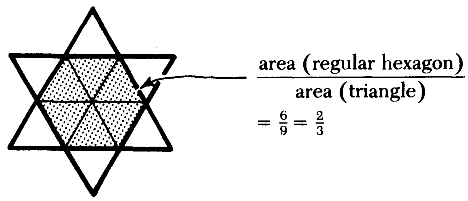


FIGURE 23

As another example, since the regular octahedron in FIGURE 15 has half the volume of the tetrahedron, the tetrahedron's measure of symmetry is $\frac{1}{2}$.

In general, the volume of a regular $n$-dimensional simplex of edgelength $(n+1)/\sqrt{2}$ is by (14)

$$\sqrt{\frac{n+1}{2^n}}\,\frac{\left(\frac{n+1}{\sqrt{2}}\right)^n}{n!}=\frac{(n+1)^{n+1/2}}{2^n\,n!},\tag{33}$$

and the largest centrally symmetric convex subset has $n$-dimensional volume given by (10) with $n$ replaced by $n+1$ and $t=(n+1)/2$:

$$\frac{\sqrt{n+1}}{2^n\,n!}\sum_{j=0}^{[(n+1)/2]}(-1)^j\binom{n+1}{j}(n+1-2j)^n.\tag{34}$$

Therefore, we see that the measure of symmetry of an $n$-dimensional regular simplex is

$$\frac{1}{(n+1)^n}\sum_{j=0}^{[(n+1)/2]}(-1)^j\binom{n+1}{j}(n+1-2j)^n.\tag{35}$$

This expression was derived by Fáry and Rédei **[5]**. The equivalent integral form, from (28) with $n$ replaced by $n+1$ and $t=(n+1)/2$, is also given there and attributed to Paul Turán.

If we are interested in how the measure of central symmetry for a regular $n$-simplex behaves when $n$ is large, we can use the fact, equivalent to (32), that $A(n/2)$ approaches $\sqrt{6/\pi}$ as $n$ approaches $\infty$. Thus, by (33) a simplex's measure of symmetry behaves like

$$\sqrt{\frac{6}{\pi}} \cdot \frac{2^n n!}{(n+1)^{(n+1)/2}}$$

for large $n$. Using Stirling's approximation for $n! \sim \sqrt{2\pi n}\,(n/e)^n$ and a little massaging, we get the large-$n$ behavior of a regular $n$-simplex's measure of symmetry:

$$\sqrt{3} \cdot \left(\frac{2}{e}\right)^{n+1}.$$

Computation addicts may check how well this approximates the exact measure of symmetry given by (35).

For any convex body K, let S be the centrally symmetric convex body of largest volume contained in K. Then the ratio of S's volume to K's volume may be used as a measure of central symmetry of K. The preceding calculations are of some interest since Fáry and Rédei conjectured that an $n$-dimensional simplex has the smallest measure of symmetry among all $n$-dimensional convex bodies. In other words, we expect (35) to give a lower bound for the measure of symmetry of any $n$-dimensional convex body. This is known to be true in case $n = 2$, but the conjecture is still open for $n \geq 3$. Thus, for example, it is not known whether every 3-dimensional convex body contains a centrally symmetric subset with half the volume.

For any reader who may have persisted with us to the end of the article and may be interested in further applications of this type of analysis, we mention a recent paper of Weissbach **[23]**, wherein he used methods similar to those of this section to prove that if C is the usual unit $n$-cube centered at the origin, and K is a unit "cross-polytope" (generalized regular octahedron) also centered at the origin, then the volume of $K \cap C$ tends to zero as $n$ approaches $\infty$.

## 12. Concluding remark

As we finally end our odyssey through cubes and lattice points, amputated simplices, numerical triangles, probabilistic slabs and arcs, and measures of central symmetry, we are gratefully indebted to the referees for their gracious advice and gentle wisdom, which contributed much toward the improvement of this article.

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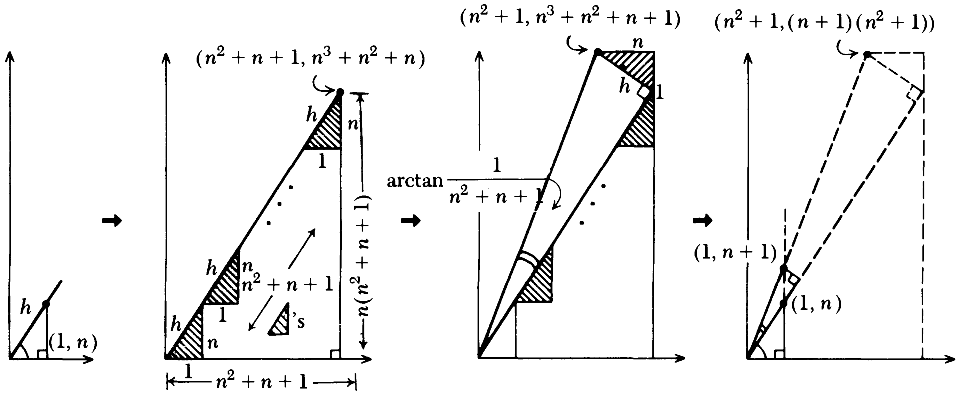
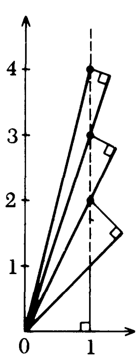
# Proof without Words: An Arctangent Identity and Series




$$\arctan n + \arctan\frac{1}{n^2+n+1} = \arctan(n+1)$$

$$\arctan\frac{1}{n^2+n+1} = \arctan(n+1) - \arctan n$$

$$\sum_{n=0}^{\infty} \arctan\frac{1}{n^2+n+1} = \lim_{N\to\infty} \arctan(N+1) = \frac{\pi}{2}$$

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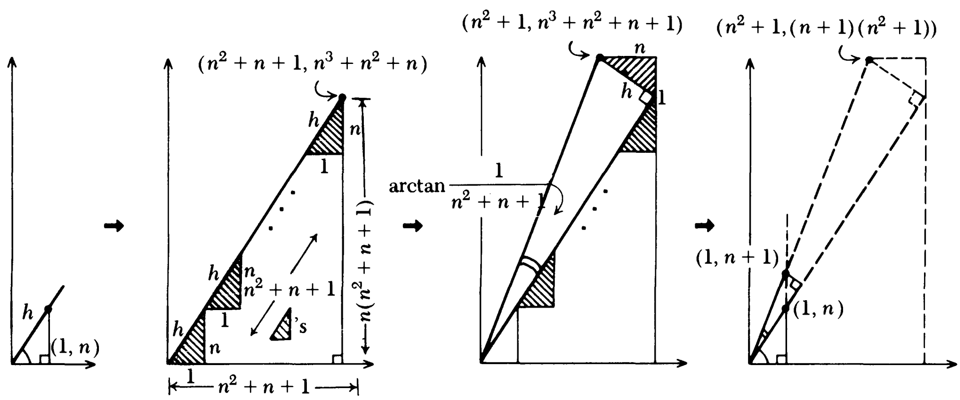


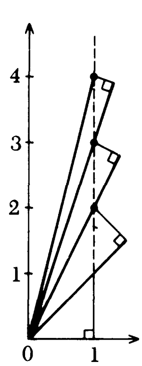


$$\arctan n + \arctan\frac{1}{n^2+n+1} = \arctan(n+1)$$

$$\arctan\frac{1}{n^2+n+1} = \arctan(n+1) - \arctan n$$

$$\sum_{n=0}^{\infty} \arctan\frac{1}{n^2+n+1} = \lim_{N\to\infty} \arctan(N+1) = \frac{\pi}{2}$$


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# NOTES

## Napoleon, Escher, and Tessellations

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Napoleon and Escher both have theorems about triangles named after them. It is doubtful whether Napoleon knew enough geometry to prove Napoleon's theorem [3, p. 63], and Escher apparently never found a proof for the last part of Escher's theorem. The first part of Escher's theorem is a form of converse of Napoleon's theorem, and both theorems can be proved using tessellations, a method that surely would have appealed to Escher with his love of filling the plane with congruent shapes.

Given any triangle, we can tessellate the plane using congruent copies of this triangle and equilateral triangles of three sizes, as shown in FIGURE 1. The centres of the small equilateral triangles in this figure clearly form the vertices of an equilateral triangular lattice, shown in FIGURE 2 by unbroken lines. The centres of the remaining equilateral triangles lie at the centres of the triangles of the lattice; hence we see from FIGURE 2 that the centres of *all* the equilateral triangles form the vertices of a smaller equilateral triangular lattice, shown by broken lines. FIGURE 3(a) forms just part of the tessellation in FIGURE 1; hence the centres of the three equilateral triangles in FIGURE 3(a) are the vertices of an equilateral triangle. This result is known as *Napoleon's theorem*; the proof just given can be found in [9], together with a figure showing that the same proof will work when the equilateral triangles are erected internally as in FIGURE 3(b) if we are prepared to extend our idea of a tessellation.

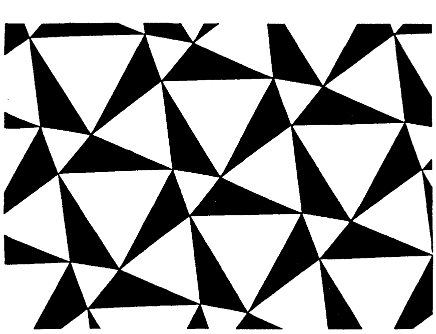

FIGURE 1

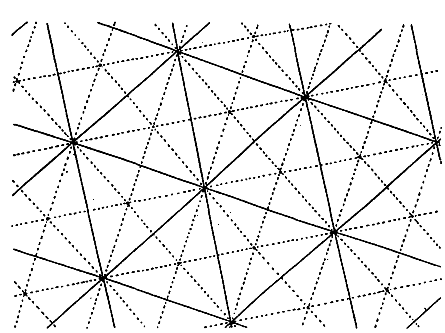

FIGURE 2

One of the notebooks of the Dutch graphic artist M. C. Escher contains some interesting results about a special type of hexagon. Although these results were known previously, we shall group them together under the title of *Escher's theorem*. The theorem may be stated as follows.

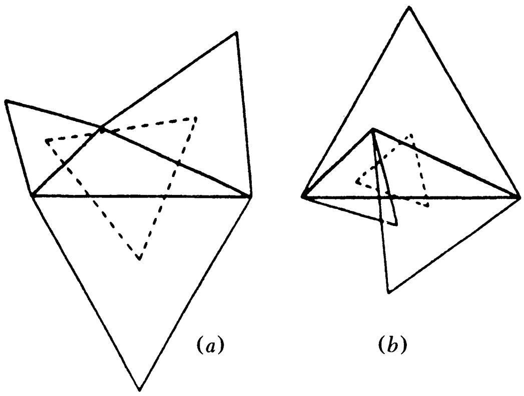


FIGURE 3

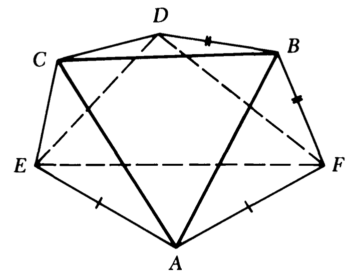


FIGURE 4

(i) *Let ABC be an equilateral triangle and E any point* (FIGURE 4). *Let F be the point such that* $AF = AE$ *and* $\angle FAE = 120°$. *Let D be the point such that* $BD = BF$ *and* $\angle DBF = 120°$. *Then* $CE = CD$ *and* $\angle ECD = 120°$.

(ii) *Congruent copies of the hexagon AFBDCE can be used to tessellate the plane.*

(iii) *In* FIGURE 4 *the lines AD, BE, and CF are concurrent.*

Escher's notebooks have been edited by Professor Doris Schattschneider [**10**]. She remarked in a letter to me that "it is very likely that Escher learned about the special tiling hexagon in a paper by F. Haag [**5**]; this paper and an earlier one [**4**] were on the list of references provided to him in 1937 by his half-brother B. G. Escher. He studied [**5**] pretty carefully, copying many diagrams, including one showing a tiling of the special hexagon. As far as I can see from the articles, Haag makes no mention of the diagonals of the hexagon. The facts are that in stating his theorem, Escher underlined his statement about the diagonals and also had no reference and no proof —that makes me pretty sure it was his own discovery. Also he wrote to his son George to ask if he could prove the result."

To prove (i) we apply Napoleon's theorem to the triangle $DEF$. The point $A$ is the centre of the equilateral triangle erected on $EF$, and $B$ is the centre of the equilateral triangle erected on $FD$. The centres of all three equilateral triangles erected on the sides of $DEF$ form an equilateral triangle by Napoleon's theorem; but $ABC$ is equilateral, so $C$ must be the centre of the equilateral triangle erected on $DE$. The result follows immediately.

The astute reader will have noticed that there are two equilateral triangles that can be erected on $EF$ and on $FD$; also there are two equilateral triangles having $AB$ as one side. Hence our proof was not sufficiently careful. But the theorem itself has not been stated carefully enough! FIGURE 5 satisfies the conditions of the theorem, and

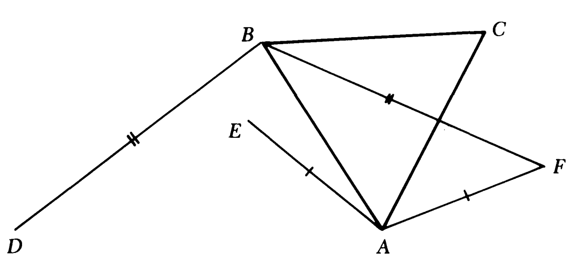


FIGURE 5

the angles $\angle FAE$ and $\angle DBF$ have even been measured in the same direction, yet the conclusion of the theorem is not valid. This is because the triangle $ABC$ has "the wrong orientation". The difficulty can be resolved by a more careful statement of the theorem, but this point presumably did not worry Escher, so we shall not let it worry us.

The tessellation in (ii) can be obtained immediately from FIGURE 1 by joining the centre of each equilateral triangle to its three vertices, as in FIGURE 6.

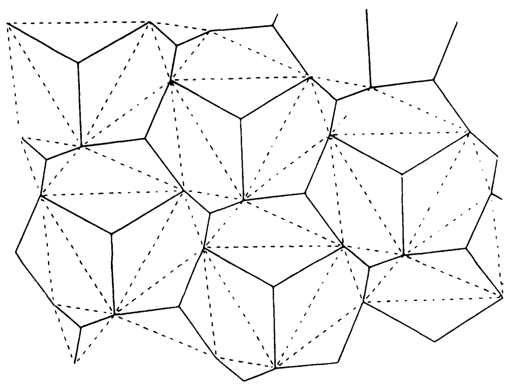

FIGURE 6

The third part of Escher's theorem is a special case of the following result.

*Let $A'EF, B'FD, C'DE$ be similar isosceles triangles erected on the sides of a triangle $DEF$, as in* FIGURE 7. *Then the lines $A'D, B'E, C'F$ are concurrent. It can also be shown that, as the shape of the isosceles triangles varies, the locus of the point of concurrency is a rectangular hyperbola passing through $D$, $E$, and $F$.*

A proof of these results can be found in [**2**]; see also [**8**]. Three other references have been supplied by Hans Cornet: Proofs of Escher's special case can be found in [**6**] and [**7**], and a proof of concurrency in the general case, from a book by O. Bottema [**1**, 1st ed. p. 36, 2nd ed. p. 51], is so short and elegant that it is worth reproducing here.

In FIGURE 7, $ED''/D''F = \triangle EDA'/\triangle FDA'$ (using $\triangle$ to denote the area of a triangle) $= \frac{1}{2}DE \cdot EA' \sin(E+\phi)/\frac{1}{2}FD \cdot FA' \sin(F+\phi) = DE\sin(E+\phi)/FD\sin(F+\phi)$. Using this and two similar expressions we find that

$$(ED''/D''F)(FE''/E''D)(DF''/F''E) = 1,$$

and the result now follows by the converse of Ceva's theorem.

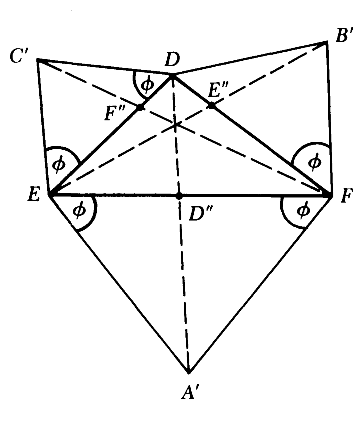


FIGURE 7

Here is another proof of Escher's special case, using rotational and translational properties of the tessellation of hexagons. Let $AD$ and $BE$ meet at $X$ (FIGURE 8). Then $C$ is the centre of the equilateral triangle $XGJ$; hence

$$JG \perp CX. \qquad (*)$$

Also $A$ is the centre of the equilateral triangle $XKY$; hence $XY \perp AK \| ZF$. Similarly $ZX \perp BL \| YF$. Hence $F$ is the orthocentre of triangle $XYZ$. Thus

$$XF \perp YZ \| JG \perp CX \text{ from } (*).$$

Hence $CXF$ is a straight line; i.e. $CF$ passes through $X$.

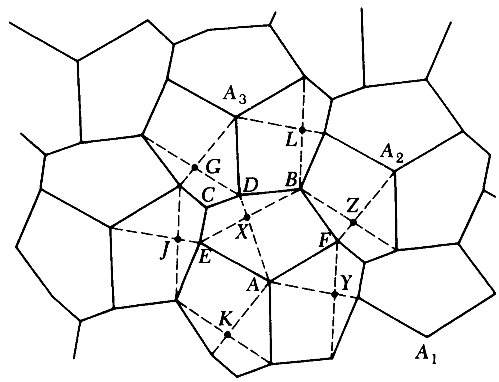


FIGURE 8

Further correspondence with Doris Schattschneider produced more items of interest. Escher's tessellation number 10 in his "abstract motif" notebook is amazingly similar to FIGURE 1, although I was not previously aware of his tessellation. FIGURE 9 shows the essential features of Escher's tessellation number 11 from the same notebook. It is built up of four congruent tessellations of hexagons, each of a different colour in the original, and the way in which the figure is built up can be described in the following way. The basic tessellation of FIGURE 8 is transformed into itself by three basic translations determined by the vectors $AA_1$, $AA_2$, and $AA_3$ (making angles of 60° with each other). If we translate the tessellation in these same three directions, but through just half the distance, we obtain the three other tessellations of FIGURE 9. (The hexagons in FIGURE 9 are of a different shape from

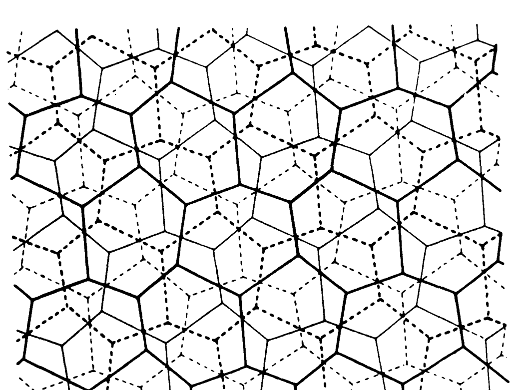

FIGURE 9

those in FIGURE 8, but this does not affect the method.) The most interesting fact about this compound tessellation is that inside any hexagon of any one tessellation there are three *concurrent* edges from the other three tessellations. Here is a short proof of this fact.

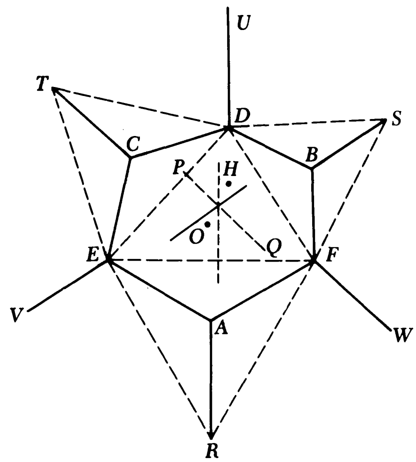


**FIGURE 10**

FIGURE 10 shows part of the original tessellation (with three of the equilateral triangles associated with it; compare FIGURE 6) and the three relevant edges of the other three tessellations. Let $O$ and $H$ be the circumcentre and orthocentre of triangle $DEF$, and let $N$ be the midpoint of $OH$, so that $N$ is the nine-point centre of $DEF$. Since $C$ is the centre of the equilateral triangle $DTE$, the line $TC$ is the perpendicular bisector of $DE$; hence $TC$ passes through $O$. Since $FW$ is parallel to $TC$, it is perpendicular to $DE$; hence $FW$ is an altitude of triangle $DEF$, and so it passes through $H$. One of the basic translations mentioned earlier transforms $TC$ to $FW$. A translation in the same direction but of half the magnitude transforms $TC$ to the edge $PQ$ of one of the other tessellations; $PQ$ lies halfway between $TC$ and $FW$, and hence $PQ$ passes through $N$ which lies half-way between $O$ and $H$. Similarly the edges of the other two tessellations shown in the figure also pass through $N$.

If we form the compound tessellation using hexagons of the shape seen in FIGURE 8, the theorem just proved is still true, but one of the edges has to be produced before it passes through the point of concurrence. It can be shown that this happens because the angle $F$ of the triangle $DEF$ in FIGURE 8 is less than 30°.

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# Power Series Expansions for Trigonometric Functions via Solutions to Initial Value Problems

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**1. Introduction** The coefficients in the power series expansion of $\sec(\phi)$, valid in an interval $|\phi| < \pi/2$, involve certain integers that are generally referred to as Euler's numbers. These numbers, denoted by $E_p$, with $p$ a nonnegative integer, may be considered as known through a recursion relation. Tables, such as those that appear in [1], give the values of some of these numbers. There is, however, no simple formula that gives $E_p$ explicitly; $E_{60}$, for example, is a positive integer containing 71 digits. As a consequence, power series expansions for functions involving products of secant functions become rather complicated and explicit formulas for the coefficients in the expansions are rather difficult to obtain. For example, a power series expansion for $\sec^2(\phi)$ could be obtained from the Cauchy product of the series expansion of $\sec(\phi)$ with itself. The coefficient of $\phi^{2n}$ could be easily determined in terms of the Euler numbers. However, expansions for higher powers of $\sec(\phi)$, such as $\sec^3(\phi)$ or $\sec^4(\phi)$, have more complicated formulas for the coefficients, though the Cauchy product will, in principle, yield these expressions. Similar remarks may be made concerning the power series expansion of $\tan(\phi)$ in an interval $|\phi| < \pi/2$. The coefficients that appear in this case involve the Bernoulli numbers, $B_p$.

In this note an alternative to the Cauchy product method is given for the determination of power series that represent products of certain trigonometric functions. The method involves repeated differentiation of a differential equation and is introduced in Section 3, which follows a summary in Section 2 of certain elementary results from calculus concerning the power series expansions of $\sec(\phi)$ and $\sec^2(\phi)$. The method discussed in Section 3 is applicable to a limited class of functions. It is, however, also a method that may lead more readily than Cauchy products to power series representations of these functions. Moreover, it is applicable as a method of obtaining power series solutions to certain initial value problems.

**2. Power series expansions of $\sec(\phi)$ and $\sec^2(\phi)$** The power series expansion of $\sec(\phi)$, valid for $|\phi| < \pi/2$, is given by

$$\sec(\phi) = \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{(2n)!} \phi^{2n}. \tag{2.1}$$

The numbers $E_{2n}$ are the Euler numbers and their values may be found using the recursion relation

$$\sum_{k=0}^{n} \binom{2n}{2k} E_{2n-2k} = 0, \tag{2.2}$$

where $E_0 = 1$.

The formula (2.2) is obtained from a Cauchy product of the series for $\sec(\phi)$ and $\cos(\phi)$. Thus,

$$\cos(\phi)\sec(\phi) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{E_{2n-2k}}{(2k)!(2n-2k)!}\right)(-1)^n\phi^{2n} = 1. \tag{2.3}$$

Since $E_0 = 1$, formula (2.2) then gives the following values for the next six Euler numbers:

$$\begin{aligned} E_2 &= -1 \\ E_4 &= 5 \\ E_6 &= -61 \\ E_8 &= 1385 \\ E_{10} &= -50521 \\ E_{12} &= 2{,}702{,}765. \end{aligned} \tag{2.4}$$

A power series expansion for $\sec^2(\phi)$ is then obtained by taking the Cauchy product of the series for $\sec(\phi)$ with itself. The result is

$$\sec^2(\phi) = 1 + \sum_{n=1}^{\infty}\left(\sum_{k=0}^{n}\binom{2n}{2k}E_{2k}E_{2n-2k}\right)\frac{(-1)^n}{(2n)!}\phi^{2n}. \tag{2.5}$$

We note that the odd order derivatives of $\sec(\phi)$ and $\sec^2(\phi)$ vanish when $\phi = 0$. The even order derivatives are given from (2.1) and (2.5) by

$$(\sec(\phi))^{(2n)}\big|_{\phi=0} = (-1)^n E_{2n} \tag{2.6a}$$

$$(\sec^2(\phi))^{(2n)}\big|_{\phi=0} = (-1)^n \sum_{k=0}^{n}\binom{2n}{2k}E_{2k}E_{2n-2k}. \tag{2.6b}$$

Hence, for example,

$$(\sec(\phi))^{(8)}\big|_{\phi=0} = E_8 = 1385,$$

while

$$\begin{aligned}(\sec^2(\phi))^{(6)}\big|_{\phi=0} &= -\left\{\binom{6}{0}E_0E_6 + \binom{6}{2}E_2E_4 + \binom{6}{4}E_4E_2 + \binom{6}{6}E_6E_0\right\} \\ &= -\{-61 + 15(-5) + 15(-5) - 61\} = 272.\end{aligned}$$

**3. Power series expansions** A power series expansion for $\sec^4(\phi)$ could be obtained by a Cauchy product of the series for $\sec^2(\phi)$ with itself. An alternative method, which we present here, relies on certain nice features of the secant function. First, let us observe that if $u(\phi) = \sec(\phi)$, then $u$ is a solution of the initial value problem

$$\begin{aligned}(u')^2 + u^2 &= \sec^4(\phi) \\ u(0) &= 1.\end{aligned} \tag{3.1}$$

If we denote evaluations of the derivatives of $u$ at $\phi = 0$ by the subscript 0, then repeated differentiation of the equation (3.1) followed by evaluations at $\phi = 0$ yields

$$u_0^{(1)}\left[u_0^{(2)} + u_0^{(0)}\right] = 2\sec^4(\phi)\tan\phi\big|_{\phi=0} = \left[\frac{\sec^4(\phi)}{2}\right]^{(1)}_{\phi=0}$$

$$u_0^{(2)}\left[u_0^{(2)}+u_0^{(0)}\right]+u_0^{(1)}\left[u_0^{(3)}+u_0^{(1)}\right]=\left[\frac{\sec^4(\phi)}{2}\right]_{\phi=0}^{(2)}$$

$$u_0^{(3)}\left[u_0^{(2)}+u_0^{(0)}\right]+2u_0^{(2)}\left[u_0^{(3)}+u_0^{(1)}\right]+u_0^{(1)}\left[u_0^{(4)}+u_0^{(2)}\right]=\left[\frac{\sec^4(\phi)}{2}\right]_{\phi=0}^{(3)}.$$

Only even order derivatives can yield nonzero results since $u_0^{(2k-1)}=0$ for every positive integer $k$. Hence a general formula for $[\sec^4(\phi)/2]_{\phi=0}^{(2n)}$ can be obtained, and we find

$$\sum_{k=0}^{n-1}\binom{2n-1}{2k}u_0^{(2n-2k)}\left[u_0^{(2k+2)}+u_0^{(2k)}\right]=\left[\frac{\sec^4(\phi)}{2}\right]_{\phi=0}^{(2n)}. \tag{3.2}$$

Since $u_0^{(2p)}=[\sec(\phi)]_{\phi=0}^{(2p)}=(-1)^pE_{2p}$, we must then have

$$\sum_{k=0}^{n-1}\binom{2n-1}{2k}(-1)^{n+1}E_{2n-2k}(E_{2k+2}-E_{2k})=\left[\frac{\sec^4(\phi)}{2}\right]_{\phi=0}^{(2n)}. \tag{3.3}$$

Thus we have obtained a power series expansion for $\sec^4(\phi)$ given by

$$\sec^4(\phi)=1+2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n)!}\left(\sum_{k=0}^{n-1}\binom{2n-1}{2k}E_{2n-2k}(E_{2k+2}-E_{2k})\right)\phi^{2n}. \tag{3.4}$$

The first few terms in the series expansion are

$$\sec^4(\phi)=1+2\phi^2+\frac{7}{3}\phi^4+\frac{94}{45}\phi^6+\frac{502}{315}\phi^8+\cdots. \tag{3.5}$$

A similar procedure could be used to obtain a power series for $\sec^6(\phi)$. In this case we observe that $u(\phi)=\sec^2(\phi)$ is a solution to the initial value problem

$$(u')^2+4u^2=4\sec^6(\phi) \tag{3.6}$$
$$u(0)=1.$$

Repeated differentiation of the equation (3.6) followed by evaluation at $\phi=0$ yields

$$\sum_{k=0}^{n-1}\binom{2n-1}{2k}u_0^{(2n-2k)}\left[u_0^{(2k+2)}+4u_0^{(2k)}\right]=\left[2\sec^6(\phi)\right]_{\phi=0}^{(2n)}, \tag{3.7}$$

where

$$u_0^{(2p)}=\left[\sec^2(\phi)\right]_{\phi=0}^{(2p)}=(-1)^p\sum_{k=0}^{p}\binom{2p}{2k}E_{2k}E_{2p-2k}. \tag{3.8}$$

Hence we obtain an expansion for $\sec^6(\phi)$ given by

$$\sec^6(\phi)=1+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n)!}\left(\sum_{k=0}^{n-1}\binom{2n-1}{2k}\left(\sum_{j=0}^{n-k}\binom{2n-2k}{2j}E_{2j}E_{2n-2k-2j}\right)\Lambda\right)\phi^{2n}, \tag{3.9}$$

where

$$\Lambda=\sum_{l=0}^{k+1}\binom{2k+2}{2l}E_{2l}E_{2k-2l+2}-4\sum_{l=0}^{k}\binom{2k}{2l}E_{2l}E_{2k-2l}. \tag{3.10}$$

The first few terms in the expansion (3.9) are given by

$$\sec^6(\phi)=1+3\phi^2+5\phi^4+\frac{92}{15}\phi^6+\cdots. \tag{3.11}$$

The success of this method hinged in part on the fact that we chose a particular differential equation whose known solution, an even function, could be repeatedly differentiated and whose derivatives at 0 were known. It is possible to devise other examples for which this method could be applied. The fact that the solutions of (3.1) and (3.6) are even functions is not crucial in the procedure of repeated differentiation of the differential equation. For example, the initial value problem

$$\begin{aligned} u''-2u&=2\tan^3\phi,\ 0<\phi<\pi/2\\ u(0)&=0\\ u'(0)&=1 \end{aligned} \tag{3.12}$$

has as its unique solution $u(\phi)=\tan\phi$, an odd function. The procedure of repeated differentiation of the differential equation may be applied in this case to obtain a series expansion for $\tan^3(\phi)$, since the derivatives of $\tan(\phi)$ at $\phi=0$ are known. Since

$$\begin{aligned}\tan(\phi)&=\sum_{n=1}^{\infty}\frac{(-1)^k2^{2k}(2^{2k}-1)}{(2k)!}B_{2k}\phi^{2k-1}\\ &=\phi+\frac{1}{3}\phi^3+\frac{2}{15}\phi^5+\frac{17}{315}\phi^7+\frac{62}{2835}\phi^9+\cdots,\end{aligned} \tag{3.13}$$

where the $B_{2k}$ are Bernoulli numbers, the first few of which are

$$\begin{aligned} B_2&=\frac{1}{6} & B_8&=-\frac{1}{30}\\ B_4&=-\frac{1}{30} & B_{10}&=\frac{5}{66}\\ B_6&=\frac{1}{42} \end{aligned} \tag{3.14}$$

we are then led to equations analogous to (3.2) and (3.3). The results are

$$u_0^{(2k+1)}-2u_0^{(2k-1)}=\left[2\tan^3(\phi)\right]_{\phi=0}^{(2k-1)} \tag{3.15}$$

and, hence,

$$\left[2\tan^3(\phi)\right]_{\phi=0}^{(2k-1)}=(-1)^k2^{2k+1}\left\{\frac{2(2^{2k+2}-1)B_{2k+2}}{(2k+2)}+\frac{(2^{2k}-1)B_{2k}}{(2k)}\right\}. \tag{3.16}$$

The series expansion for $\tan^3(\phi)$ is then given by

$$\tan^3(\phi)=\sum_{k=2}^{\infty}\frac{(-1)^k2^{2k}}{(2k-1)!}\left[\frac{2(2^{2k+2}-1)B_{2k+2}}{(2k+2)}+\frac{(2^{2k}-1)B_{2k}}{(2k)}\right]\phi^{2k-1} \tag{3.17}$$

$$=\phi^3+\phi^5+\frac{11}{15}\phi^6+\frac{88}{189}\phi^9+\cdots. \tag{3.18}$$

Still another example, from which an expansion for $\sec^3(\phi)$ may be found, is provided by the initial value problem

$$\begin{aligned} u'' + u &= \left[2\sec^3\phi\right] \\ u(0) &= 1 \\ u'(0) &= 0. \end{aligned} \tag{3.19}$$

It is easily verified that $u(\phi) = \sec(\phi)$ is a solution. The procedure of repeatedly differentiating the differential equation then results in the expansion

$$\begin{aligned} \sec^3(\phi) &= \sum_{k=0}^{\infty} \frac{(-1)^{k-1}}{2(2k)!}[E_{2k+2} - E_{2k}]\phi^{2k} \\ &= 1 + \frac{3}{2}\phi^2 + \frac{11}{8}\phi^4 + \cdots, \end{aligned} \tag{3.20}$$

valid on the interval $|\phi| < \pi/2$.

The expansions obtained in (3.4), (3.17), and (3.20) are simpler in form than those that would have been obtained by taking Cauchy products. On the other hand, the process by which these series were obtained is somewhat special in that the solution to an initial value problem must be known, along with the values of the derivatives of the solution at 0. Thus the initial value problem has to be carefully chosen. We note also that the process of repeated differentiation of a differential equation may sometimes be used to generate coefficients for series solutions. The reader interested in this application should see **[3]**, for example.

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"It always seems to me absurd to speak of a complete proof, or of a theorem being rigorously demonstrated. An incomplete proof is no proof, and a mathematical truth not rigorously demonstrated is not demonstrated at all."

—*Sylvester's Collected Works*, Vol. 2, p. 200

# Tree Isomorphism Algorithms: Speed vs. Clarity

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No algorithm, other than brute force, is known for testing whether two arbitrary graphs are isomorphic. In fact it is still an open question [3] whether graph isomorphism is NP complete. But polynomial time isomorphism algorithms for various graph subclasses such as trees are known (see [3, p. 285 and p. 339] for a summary). In gene splicing, protein analysis, and molecular biology the chemical structures are often trees with millions of vertices. In such applications, the difference between $O(n)$, $O(n \log n)$, and $O(n^2)$ isomorphism algorithms is of practical not just theoretical importance. Readers of MATHEMATICS MAGAZINE will find it of interest to see how such algorithms evolve and are analyzed. As in an earlier article [2], we have chosen dialogue to capture the spirit of discovery and failure in the search for a fast and clear tree isomorphism algorithm.

*Jill.* I need a quick way to determine whether two trees, such as those of FIGURE 1, are isomorphic.


**FIGURE 1**

*Peter.* What do you mean! They are isomorphic under the trivial correspondence $A$–$a$, $B$–$b$, $C$–$c$, $D$–$d$.

*Jill.* Well, yes and no. That is a *graph* isomorphism. Formally, if $E$ is a set of edges, $V$ a set of vertices, and $r$ a distinguished vertex called the root, then a *tree* $T = (E, V, r)$ is a connected, acyclic graph. A *tree isomorphism* must match roots, as well as be a graph isomorphism.

*Peter.* Providing a quick way to *see* a tree may provide a quick way to determine whether two trees are isomorphic. Assuming each edge of a tree has length one, let's assign to each vertex its distance to the root. This distance is the *level number of the vertex*. The root's level number is zero. Draw the tree, level by level, starting with the root at the top. A vertex with no descendants is a *leaf*. Allow me an observation.

*Observation* 1. *Since a tree isomorphism preserves root and edge incidence, the level number of a vertex (the number of edges between the root and the vertex) is a tree isomorphism invariant.* Here's the quick test you wanted:

CONJECTURE 1. *Two trees are isomorphic if and only if they have the same number of levels and the same number of vertices on each level.*

An array implementation is simple and fast. Start with an array $L[0..\|V\|]$ whose entries have been set to zero. Let $L[n]$ denote the number of vertices of level $n$. For each vertex, if the vertex has level number $n$, then increment $L[n]$ by one.

Conjecture 1 says we need only test if two arrays are equal, which takes time proportional to $\|V\|$.

*Jill.* You haven't added in the time it takes to determine the level numbers, nor the time to fill the array, but never mind. Your conjecture is false. For every $\|V\| > 4$, I can construct two non-isomorphic trees that have $\|V\|$ vertices, the same number of levels, and the same number of vertices at each level. Let me make an observation.

*Observation* 2. *Since a tree isomorphism preserves root and edge incidence, the number of paths from the root to the leaves is a tree isomorphism invariant.*

With observation 2 in mind, let $T_1$ be the tree of FIGURE 2 with $n-3$ vertices at level one, one of which has two descendants. Let $T_2$ be the tree of FIGURE 2 with $n-3$ vertices at level one, two of which have one descendant. Since $T_1$ has $n-2$ paths from root to leaves and $T_2$ has $n-3$ paths from root to leaves, they cannot be isomorphic.

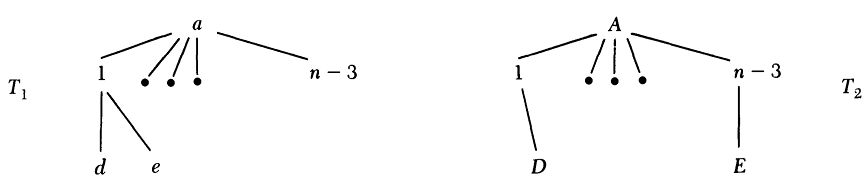


FIGURE 2

You didn't take into account the *degree spectrum of a tree*, the sequence of non-negative integers $(d_j)$, where $d_j$ is the number of vertices in $T$ that have $j$ descending edges when the tree is drawn from the root down. I conjecture:

CONJECTURE 2. *Two trees are isomorphic if and only if they have the same degree spectrum.*

An array implementation is simple and fast. Start with an array $D[0..\|V\|]$ whose entries are set to zero. Let $D[n]$ denote the number of vertices with $n$ descending edges. For each vertex, if the vertex has $n$ descending edges, increment $D[n]$ by one. Again, we are reduced to testing the equality of two arrays of size $\|V\|$, an operation requiring $2\|V\|$ time.

*Peter.* This time you failed to add the time it takes to compute the degree of each vertex. No point in doing that since your conjecture is false. Let me make an observation.

*Observation* 3. *Since a tree isomorphism preserves longest paths, the number of levels in a tree (the longest path) is a tree isomorphism invariant.*

With observation 3 in mind, I can construct two non-isomorphic trees of $n$ vertices, $n > 6$, that have the same degree spectrum. Let $T_1$ be the tree of FIGURE 3 with a strand of $n-5$ segments from $C$. Let $T_2$ be the tree of FIGURE 3 with a strand of $n-5$ segments from $b$. Both have degree spectrum $(3, n-5, 2)$. But $T_1$'s longest path has length $n-3$, while $T_2$'s longest path has length $n-4$.

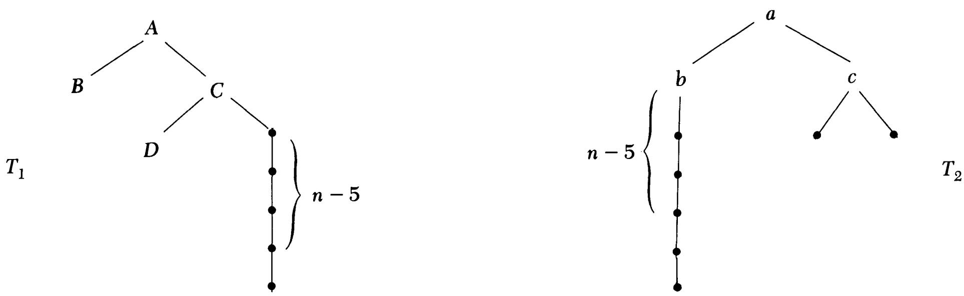


FIGURE 3

However, I conjecture:

CONJECTURE 3. *Two trees are isomorphic if and only if they have the same degree spectrum at each level.*

*Jill.* Tricky. If two trees have the same degree spectrum at each level, then they must automatically have the same number of levels, the same number of vertices at each level, and the same global degree spectrum!

Nevertheless, Conjecture 3 is false for all $n > 7$. Let me make an observation.

*Observation* 4. *The number of leaf descendants of a vertex and the level number of a vertex are both tree isomorphism invariants.*

With observation 4 in mind, I can construct two non-isomorphic trees of $m+8$ vertices, $m > 0$, whose degree spectrums are equal level by level. Indeed $T_1$ and $T_2$ of FIGURE 4 cannot be isomorphic since the number of leaf descendants of $A$, $B$, $C$, and $D$ on level one are, respectively, 2, 2, 3, 1.

We must measure not only the degree spectrum at each level, but also the degree spectrum of each vertex's descendants.

*Peter.* Enough of this. Let's quote the AHU algorithm by Aho, Hopcroft, and Ullman [1] that purports to determine tree isomorphism in time $O(\|V\|)$ by keeping track of the history of the degree spectrum of the descendants of a vertex. The algorithm assigns numbers to the vertices of trees in such a way that trees $T_1$ and $T_2$ are isomorphic if and only if the same number is assigned to the root of each tree.

1. Assign to all leaves of $T_1$ and $T_2$ the integer 0.

2. Inductively, assume that all vertices of $T_1$ and $T_2$ at level $i-1$ have been assigned integers. Assume $L_1$ is a list of the vertices of $T_1$ at level $i-1$ sorted by non-decreasing value of the assigned integers. Assume $L_2$ is the corresponding list for $T_2$.

3. Assign to the non-leaves of $T_1$ at level $i$ a tuple of integers by scanning the list $L_1$ from left to right and performing the following actions: For each vertex $v$ on list $L_1$ take the integer assigned to $v$ to be the next component of the tuple associated with the father of $v$. On completion of this step, each non-leaf $w$ of $T_1$ at level $i$ will have a tuple $(i_1, i_2, \ldots, i_k)$ associated with it, where $i_1, i_2, \ldots, i_k$ are the integers, in

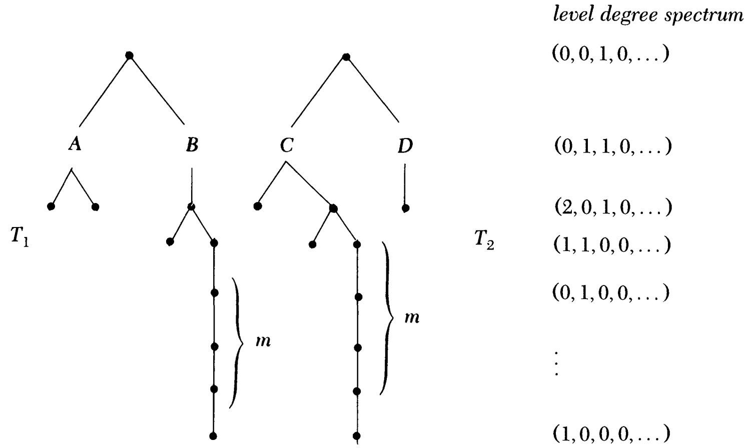


**FIGURE 4**

non-decreasing order, associated with the sons of $w$. Let $S_1$ be the sequence of tuples created for the vertices of $T_1$ on level $i$.

4. Repeat step 3 for $T_2$ and let $S_2$ be the sequence of tuples created for the vertices of $T_2$ on level $i$.

5. Sort $S_1$ and $S_2$ lexicographically. Let $S_1'$ and $S_2'$ respectively, be the sorted sequences of tuples.

6. If $S_{1'}$ and $S_{2'}$ are not identical then halt; the trees are not isomorphic. Otherwise, assign the integer 1 to those vertices of $T_1$ on level $i$ represented by the first distinct tuple on $S_1'$, assign the integer 2 to the vertices represented by the second distinct tuple, and so on. As these integers are assigned to the vertices of $T_1$ on level $i$, make a list $L_1$ of the vertices so assigned. Append to the front of $L_1$ all leaves of $T_1$ on level $i$. Let $L_2$ be the corresponding list of vertices of $T_2$. These two lists can now be used for the assignment of tuples to vertices at level $i+1$ by returning to step 3.

7. If the roots of $T_1$ and $T_2$ are assigned the same integer, $T_1$ and $T_2$ are isomorphic.

*Jill*. Utterly opaque. Even on second or third reading. When an algorithm is written it should be clear, it should persuade, and it should lend itself to analysis.

Let's write an algorithm that is close to obvious, that lends itself to timing analysis, and that is clear. After we do those three things, *then* let's try to make it fast.

*Peter*. Perhaps we should begin by trying to understand the AHU algorithm.

*Jill*. The AHU algorithm associates with each vertex a tuple that describes the complete history of its descendants. In order to do that it uses a unique integer as a sort of shorthand. Since the unique integers are introduced for speed and not clarity, let's return to a simpler tuple version.

*Peter*. Let's begin by converting the tree of FIGURE 5 to the Knuth tupled tree [1] of FIGURE 6 using AHU's steps (1), (2), and (3). First, eyeball the tree and find the leaves $D$, $E$, $F$, and $G$. Assign them the tuple (0). Then fill in the vertices of level 1. Then fill in the root.

Unfortunately, the assignment has proceeded in a haphazard manner that appears to require multiple eyeball passes through the tree. Much worse, I have been forced to access the tree 'level by level' although the tree is generally stored 'parent-to-child'.

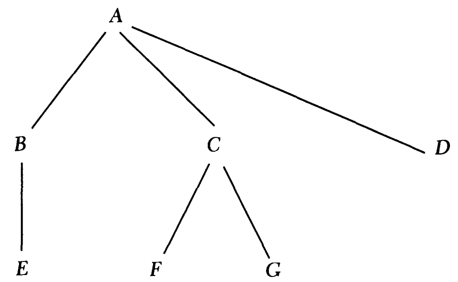


FIGURE 5

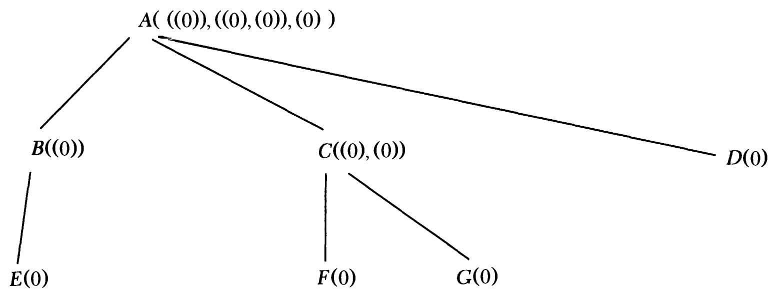


FIGURE 6

In fact, since we have both been loose on how the tree is stored, let's fix the realm of discourse. A tree *is* the root and the set of parent-child relationships. That is, if you give a tree black box the name of a vertex, it will respond with the names of the vertex's children, but cannot give the vertex's cousins, siblings, or parent! If you want to build and store *that* information, be my guest. But the time it takes to acquire that knowledge must be accounted for when analyzing the tree isomorphism algorithm.

A glance at the tree of FIGURE 5 shows that it reduces to the following facts: root $A$; $A:B, C, D$; $B:E$; $C:F, G$; $D:$ ; $E:$ ; $F:$ ; $G:$ ;. and nothing more!

*Jill.* That is an excellent point. Let me first prove that I can assign Knuth parenthetical tuples to all tree vertices in time proportional to $\|V\|$, whether the tree is a long, single strand or short and squat with multitudes of edges spilling off each vertex.

I claim the existence of a set of instructions whose execution causes every vertex of the tree to be visited exactly twice (Post Order Traversal). It is on the second visit to a vertex that the tuple name is assigned. The second visit to a vertex is made *after* all of the vertex's descendants have had their tuple name assigned. Here are the instructions.

```
Post_Order_Version_One(v: vertex);
Begin
    if v is childless then
      Give v the tuple name (0)
    else
     begin
      For each child w of v do
              Post_Order_Version_One(w);
```

```
      Concatenate the names of all the children of v to temp;
      Give v the tuple name (temp);
    end
end;
```

*Peter*. Aha! Recursion! How slick. A vertex doesn't need to know its level, only its children. If it has no children, it is automatically given the tuple name (0). If it has children, then after all of its children have been named, the children's names are concatenated, the concatenated name is enclosed in a pair of parentheses and assigned to the vertex.

*Jill*. Now that you understand this *simple* way to name vertices, I must explain why it is insufficient for an isomorphism test. Although the two trees of FIGURE 7 are isomorphic, the roots have been assigned the different names ( (0) ((0)) ) and ( ((0)) (0) ) because we didn't insist on concatenating the children's names in some fixed order.


FIGURE 7

*Peter*. But how can you order children's names? Children's names have no order! Children can be stored in any order. What rhyme or reason should make us order (0)(0) before or after ((0))?

*Jill*. Easier than you think. First, drop the 0's from names; they are superfluous. Each name becomes just a string of ('s and )'s. Interpreting each '(' as a 1 and each ')' as a 0 lets us interpret each string of ('s and )'s either as a *name* or as a *binary number*. Interpreting a string as a number lets us use the ordinary ordering for binary numbers!

*Peter*. Although your name's number isn't *natural*, I see that it does allow an ordering. No doubt you will use this ordering as follows. Take a vertex's children, order the children's names, concatenate the names in order, put a pair of parentheses around the concatenated names, and assign this new name to the vertex. Doesn't the cost of putting the names in order before they are concatenated significantly add to the cost of executing Post_Order_Version_One?

*Jill*. Yes. First let's present the modified version.

```
Post_Order_Version_Two(v: vertex);
Begin
  if v is childless then
    Give v the tuple name "10"
  else
  begin
    For each child w of v do
        Post_Order_Version_Two(w);
```

Sort the names of the children of $v$;
Set temp to the concatenation of $v$'s sorted children's names;
Give $v$ the tuple name "1*temp*0";
end
end;

In [1, p. 83] it is proved that if $A_1, A_2, \dots, A_m$ are *strings* of 0's and 1's and $l_i$ denotes the *length* of the $i$th string, then a lexicographical sort can be done in time proportional to $\Sigma l_i$, that is, proportionate to the combined lengths of all the strings. Thus summing the sort time at each vertex we see that the sort time is proportional to the length of the sum of the vertex names throughout the tree.

*Peter*. That is going to produce a very crude estimate. Since there are $n$ nodes in a tree, each with length at most $2n$, the total time is bounded by a multiple of $n^2$.

*Jill*. Unfortunately, it is *not* a crude estimate. Consider the time it takes to assign the name to the root of a tree of $n$ vertices in one long strand. The names are 10, 1100, 111000, ... with the name of the root consisting of $n$ 1's followed by $n$ 0's. The time to process the name on the $i$-th level from the bottom is proportional to $i$. Therefore to compute the *name* of the root, takes time proportional to

$$1 + 2 + \cdots + i + \cdots + n,$$

which is $O(n^2)$.

*Peter*. Let me make two observations.

*Observation* 5. *Induction on the level number proves that a vertex's canonical name is a tree isomorphism invariant.*

*Observation* 6. *Two trees are isomorphic if and only if their roots have identical canonical names.*

In summary, the Post_Order_Version_Two algorithm is intuitively sound, and for infinitely many cases takes $O(n^2)$ time.

Post_Order_Version_Two tests for isomorphism by computing the canonical name of each root node, that is, the canonical name of the 0-th level. Let's modify the algorithm so that it can detect failure early. We introduce the *canonical level name* (rather than the canonical name of a single vertex).

Let me first prove that it is straightforward and takes time proportionate to $\|V\|$ to find all vertices of level $i$ for all values of $i$. Indeed, it takes $O(\|V\|)$ time to give each vertex its ordinary tuple name with Post_Order_Version_One. While traversing the tree, simply keep track of the current distance to the root (the level number). Therefore, letting $LL[i]$, $i = 0, \dots, \|V\|$, denote the *LevelList* of all vertices on level $i$, we can simply add the vertex to its appropriate level list at the same time it receives its ordinary name.

The *canonical level name* is formed by first arranging in order the canonical names of the vertices of that level and then concatenating those names.

*Jill*. Two observations.

*Observation* 7. *For all levels $i$, the canonical name of level $i$ is a tree isomorphism invariant.*

*Observation* 8. *Two trees $T_1$ and $T_2$ are isomorphic if and only if for all levels $i$, the canonical level names of $T_1$ and $T_2$ are identical.*

These observations let us test for early failure. Instead of assigning names by Post_Order_Version_Two, we invest a pre-processing time of $O(\|V\|)$ and create the LevelLists $LL[i]$ of all vertices on level $i$. We then assign canonical names by level, sort by level, and check by level that the *canonical level names agree*. If at any level, the canonical level names disagree, then we stop since the trees are not isomorphic. Although this catches early failures, it still requires sorting and, if the failure does not occur till the last level, it may still take $O(\|V\|^2)$ time. Nevertheless, it does not have to process all the way to the root to detect failure.

*Peter.* Let me suggest another modification. The long canonical vertex names built by this algorithm contain unnecessary information, namely a recursive encryption of the complete genealogy of all of the descendants of the vertex. This complete encryption of information is propagated all the way to the root. Reading these increasingly lengthy genealogies creates the $\|V\|^2$ time bound.

I claim that keeping this complete *history* serves no useful purpose for the question of whether two trees are isomorphic. Consider the trees in FIGURE 8 that have billions of vertices. Suppose after computing the first 500 million vertices we find the trees have identical canonical names at level $j$. The canonical names of $a$ and $A$ are hundreds of millions of characters long and *identically* summarize all of the previous genealogy. But for what purpose is this information kept! Why *keep* this genealogy encrypted in a name hundreds of millions of characters long? All that the algorithm needs to know is the fact that the names are the same—the actual names aren't needed! It suffices to give $a$ and $A$ new, short names that indicate that the canonical names (whatever they actually are) match up to this point!

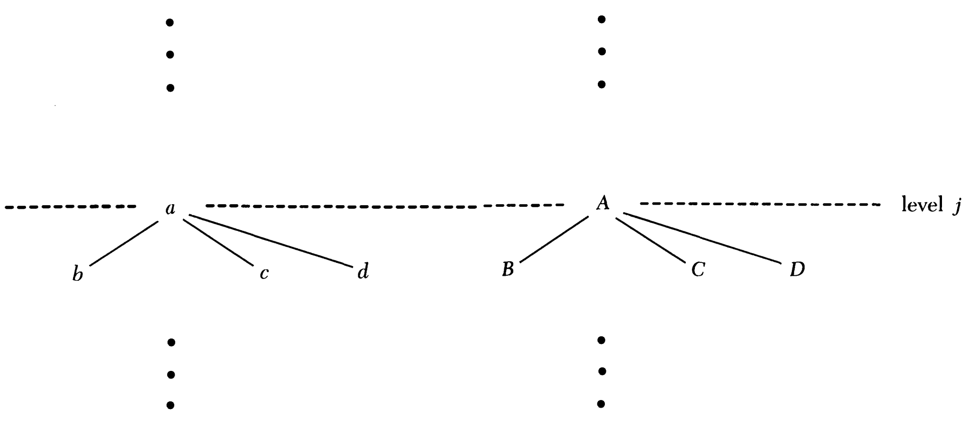


FIGURE 8

*Jill.* Sure. We could just call them both 1 and refer to them as *condensed* canonical names. Say, this is getting to look like step 6 of the AHU algorithm.

*Peter.* So how long will it take to determine isomorphism?

*Jill.* It depends. If you assume that the number of vertices in the tree is small enough to fit in a fixed computer word of $k$ bits (i.e., on a 32-bit machine, trees have no more than $2^{32}$ nodes), then AHU prove that the bound is $O(\|V\|)$.

On the other hand, if you want a bound that works even when the number of nodes in the tree is so large that numbers must be treated as *strings* of 0's and 1's, then there are trees for which it takes time $O(\|V\|\log_2\|V\|)$ to calculate the *condensed* canonical name of the root.

Consider the tree of FIGURE 9 with $N = n^2 + n(n+1)/2$ nodes. The tree in FIGURE 9 consists of strands $s_i, i = 1, \ldots, n$, where the strand $s_i$ has $n + i$ vertices. At level $2n$, the condensed canonical name is 1. At level $2n-1$, the two condensed canonical names are 1 and 2. At level $n$, the condensed canonical names $1, \ldots, n$ are required. From level $n-1$ to level 1, no additional canonical names are introduced or deleted. Therefore, the vertices on level 1 to level $n$ have names whose *lengths* go from $\log_2 1$ to $\log_2 n$. Lexicographically sorting this list at level $i$, $1 \le i \le n$, takes time proportional to the sum of the lengths,

$$\sum_{i=1}^{n} \log_2 i,$$

which is $O(n\log_2 n)$. Since this sorting is repeated $n$ times, once for each $i$, $1 \le i \le n$, the time taken is at least $O(n^2\log_2 n) = O(N\log_2 N)$ as claimed.

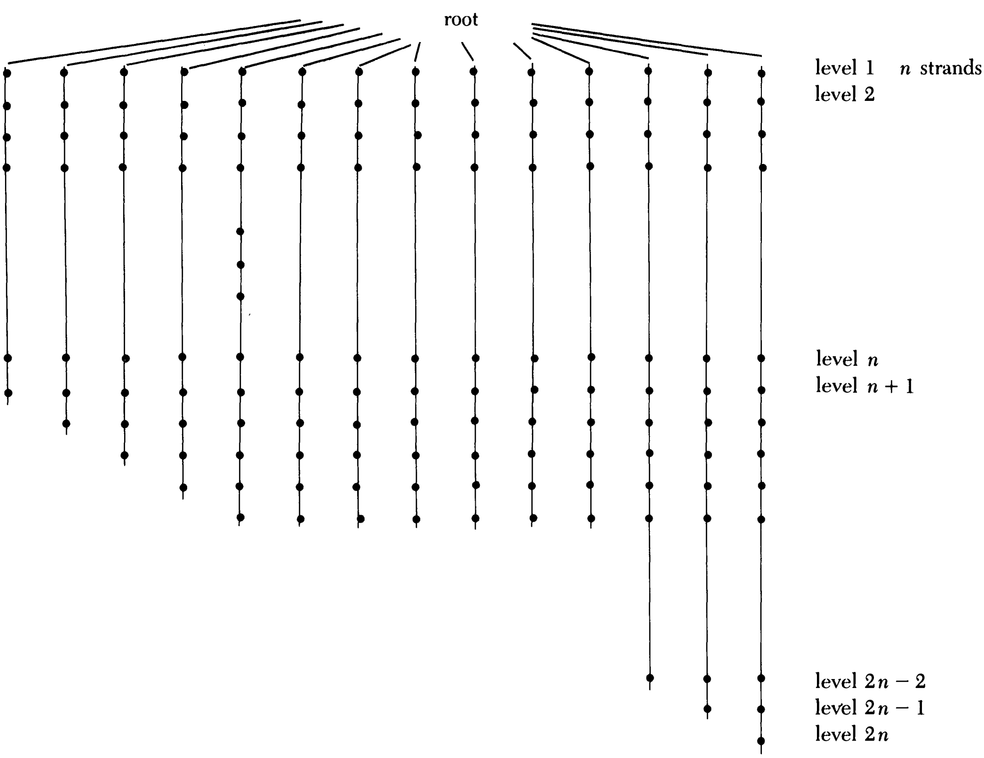


**FIGURE 9**

*Peter.* FIGURE 10 summarizes a $O(\|V\|\log_2\|V\|)$ tree isomorphism algorithm, incorporating provisions for detecting failure early, and the use of condensed canonical names.

```
Tree Isomorphism (T1, T2: trees);
Begin
    Assign all vertices of T1 and T2 to level numbers lists and let hi be the largest level number in Ti;

    If h1<>h2 then
      write ('trees are not isomorphic'); Halt;
    else
      set h to h1; { h1 = h2 }

    { process from bottom to top level }
    for i: = h downto 0 do
    begin
      { assign vertices their string name }
      For all vertices v of level i do
        If v is a leaf then
          assign v the string 10
        Else
          assign v the tuple 1 i1 i2 ... ik 0, where i1, i2, ..., ik are the strings associated with the
          children of v, in non-decreasing order;
        { assign vertices to temporary sorting lists }
        For all vertices v of level i do
          If v belongs to Tj then
            add v's string to Tj(i);
        Sort T1(i) and T2(i) lexicographically;
        If T1(i)<>T2(i) then
            write ('trees are not isomorphic at level', i); Halt;

        { assign condensed canonical names }
        For all vertices v of level i do
            If v is the k-th element in Tj(i) then
              assign v the binary string for the integer k;

    end;
    write ('the trees are isomorphic')
  end.
```

**FIGURE 10**

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2. Douglas M. Campbell, The computation of Catalan numbers, this MAGAZINE 57 (1984), 195–208.
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# Archimedes' Method for the Reflections on the Ellipse

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For an ellipse the sum of the distances from the two focal points is constant, and for the hyperbola the difference of the distances is constant. These definitions permit a short and transparent construction of the tangents to these curves, including the reflection property for the ellipse and the hyperbola [1].

We start with the mechanical device shown in FIGURE 1. Two rigid rods can turn freely around the pivots $P_1$ and $P_2$. They have a slit in which a point $M$ can move in any way, tracing out a curve $C$. The displacement $\Delta s$ along the curve $C$ is shown in FIGURE 2, together with the two circles $C_1$ and $C_2$ where $l_1 + \Delta l_1$, and $l_2 + \Delta l_2$ are constant. As $\Delta s$ goes to zero, the segment $MQ$ approaches the direction of the tangent line to $C$ at the point $M$, and the arcs $PQ$ and $RQ$ approach straight lines.

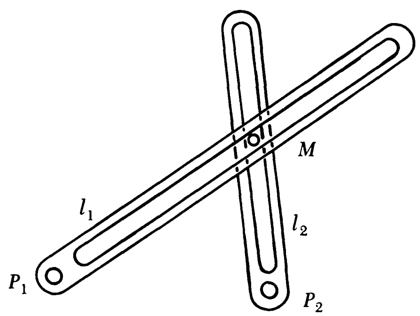


FIGURE 1

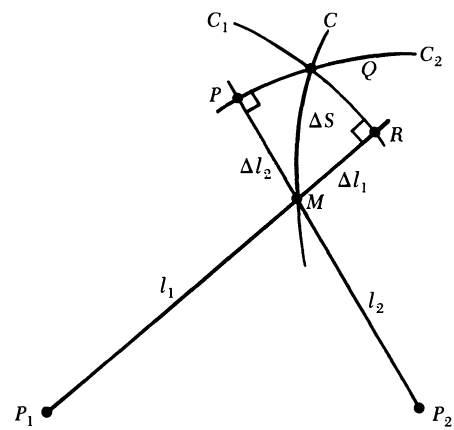


FIGURE 2

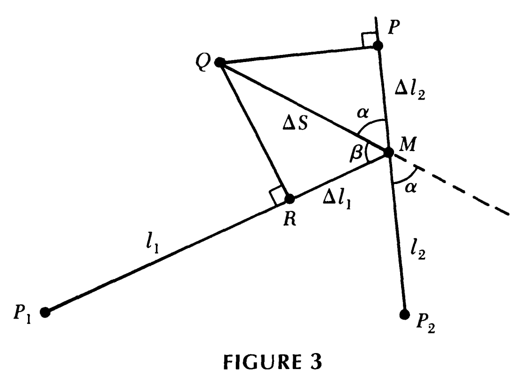


FIGURE 3

If $M$ moves on an ellipse where

$$l_1 + l_2 = \text{constant},$$

then

$$\Delta l_1 = -\Delta l_2.$$

As $\Delta s$ goes to zero, $PQM$ and $RQM$ tend to right triangles with the common base $MQ$. They are congruent and hence $\alpha = \beta$ (FIGURE 3).

The argument and the conclusion for the hyperbola are the same, except that from

$$l_1 - l_2 = \text{constant},$$
$$\Delta l_1 = \Delta l_2$$

follows.

It is well known that Archimedes (287–212 B.C.) could deal with tangents, for instance for the spiral, and that he liked to use mechanical ideas to guide his mathematical derivations. Whether he actually solved the reflection problem for the ellipse may be in question, but the solution presented here seems to be in his spirit.

## REFERENCE

1. W. C. Schulz and C. G. Moore, Reflections on the ellipse, this MAGAZINE, 60 (1987), 167.

# Generating Polynomials All of Whose Roots Are Real

G. G. BILODEAU
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Chestnut Hill, MA 02167

**1. Introduction** Polynomials with real coefficients all of whose roots are real occupy a special place in analysis, good examples being the classes of polynomials orthogonal on intervals. The purpose of this paper is to exploit extensions of some basic principles and obtain sophisticated conclusions about polynomials with only real roots.

The roots $\beta_1, \dots, \beta_n$ of a polynomial $q(x)$ of degree $n$, $n \geqslant 1$, are said to *separate* the roots, $\alpha_1 \leqslant \cdots \leqslant \alpha_{n+1}$, of $p(x)$ of degree $n+1$ if $\alpha_i \leqslant \beta_i \leqslant \alpha_{i+1}$, $i = 1, \dots, n$. Note that the definition can handle multiple roots.

Between any two real distinct roots of a real polynomial $p(x)$ there is a root of its derivative $p'(x)$. This is an immediate consequence of Rolle's theorem although it can be proved more directly. If we add to this the result that a root of $p(x)$ of multiplicity $m$ ($\geqslant 2$) is a root of $p'(x)$ of multiplicity $m-1$, a consequence of differentiation, we obtain the following:

If $M$ moves on an ellipse where

$$l_1 + l_2 = \text{constant},$$

then

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Between any two real distinct roots of a real polynomial $p(x)$ there is a root of its derivative $p'(x)$. This is an immediate consequence of Rolle's theorem although it can be proved more directly. If we add to this the result that a root of $p(x)$ of multiplicity $m$ ($\geqslant 2$) is a root of $p'(x)$ of multiplicity $m-1$, a consequence of differentiation, we obtain the following:

Proposition 1. *If $p(x)$ is a polynomial of degree at least 2 with real coefficients and only real roots, then $p'(x)$ also has only real roots and they separate the roots of $p(x)$.*

This elementary result is the point of departure for this paper. Define **P** as the class of all polynomials with real coefficients all of whose roots are real. Included in **P** are the constant polynomials. The main assertion of proposition 1 is then contained in the concise statement:

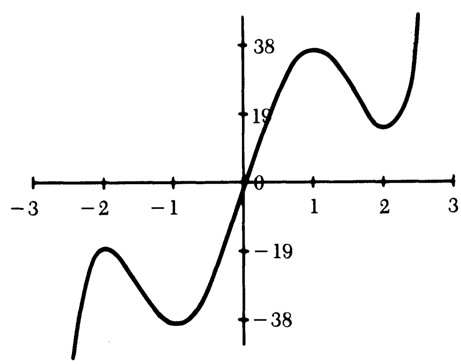
Proposition 2. $D(\mathbf{P}) \subset \mathbf{P}$.

Here $D$ stands for the operation of differentiation and $D(\mathbf{P})$ is the set of all polynomials of the form $Dp(x) = p'(x)$ with $p(x)$ in **P**.

In the present paper this result is extended by using more complicated functions of the operator $D$ leading to differential operators that are generated by a special class of entire functions. Results will be illustrated with examples. I have tried to keep the arguments elementary although more elaborate approaches are needed occasionally and will be used without detailed proofs.

While the presentation and point of view are believed to be new, most of the results are old and have a long history. To avoid interrupting the development with references, a careful citation of sources has been postponed to the last section.

**2. Polynomial operators** The containment in proposition 2 is strict. It is elementary but oddly complicated to show this. Let $p(x) = 3x^5 - 25x^3 + 60x$. Then $p'(x) = 15(x^2 - 1)(x^2 - 4)$ and thus is in **P**. We will show that there is no polynomial in **P** whose derivative is $p'(x)$. A simple analysis shows that $p(x)$ has local maxima at $(1, 38)$ and $(-2, -16)$ and local minima at $(2, 16)$ and $(-1, -38)$. A sketch of the graph of $p(x)$ (see below) makes it clear that $p(x) + c$, $c$ any constant, (the form of all antiderivatives of $p'(x)$) cannot have 5 real roots for any $c$ and hence cannot be in **P**.




It is curious that a polynomial of degree at least 4 (for $p'(x)$) is needed to show this.* We now extend proposition 2 to

Proposition 3. $(D - \lambda)\mathbf{P} \subset \mathbf{P}$ *for any real number* $\lambda$.

$(D - \lambda)\mathbf{P}$ is the set of all polynomials of the form $(D - \lambda)p(x) = p'(x) - \lambda p(x)$ with $p(x)$ in **P**. If $p(x)$ in **P** is a constant, the result is clear. Otherwise the degree of $p(x)$ is $n \geqslant 1$, in which case let $\alpha_1, \ldots, \alpha_k$ be its distinct roots with multiplicities $m_1, \ldots, m_k$, respectively, so that $\sum_1^k m_i = n$ and

*Communicated to the author by Stephen J. Ricci.

$$p(x) = c\prod_{1}^{k}(x-\alpha_i)^{m_i}, \qquad c \neq 0. \tag{2.1}$$

Then for $\lambda \neq 0$,

$$(D-\lambda)p(x) = p'(x) - \lambda p(x) = e^{\lambda x}D\left[e^{-\lambda x}p(x)\right] = q(x), \tag{2.2}$$

and we conclude by differentiating that $q(x)$ is a polynomial of degree $n$ with roots at $\alpha_1, \dots, \alpha_k$ of multiplicities $m_1 - 1, \dots, m_k - 1$ respectively, and by Rolle's theorem has at least $k-1$ roots corresponding to the intervals $(\alpha_i, \alpha_{i+1})$, $i = 1, \dots, k-1$. The number of real roots is then at least $\{[\Sigma_1^k(m_i - 1)] + (k-1)\} = (n-k) + (k-1) = n-1$. It follows that all $n$ roots of $q(x)$ are real since complex roots must occur in conjugate pairs for polynomials with real coefficients. This argument assumes that $k$ (and hence $n$) is at least 2. When $k = 1$, $q(x) = c(x-\alpha_1)^{m_1 - 1}[m_1 - \lambda(x - \alpha_1)]$ and the result is again clear. This proves the proposition. It will be useful later to observe that $n-1$ of the roots of $q(x)$ separate the roots of $p(x)$ when $n \geqslant 2$.

The converse of proposition 3 is not true. This was shown earlier when $\lambda = 0$; but the argument when $\lambda \neq 0$ is simpler. For let $p(x) = x^2$, an element of **P**, then the polynomial solution of $(D-\lambda)q(x) = p(x)$ is $q(x) = -\lambda^{-3}(\lambda^2x^2 + 2\lambda x + 2)$, which has complex roots for any $\lambda \neq 0$ and thus is not in **P**.

It now follows by induction from proposition 3 that a repetition of operators of the form $(D-\lambda)$ on elements of **P** will lead to polynomials in **P**. This can be elegantly stated as:

PROPOSITION 4. *$p(D)\mathbf{P} \subset \mathbf{P}$ for any $p(x)$ in **P**.*

We understand by $p(D)$ the operator obtained from (2.1) with $x$ replaced by $D$ when the degree of $p(x)$ is at least one; otherwise $p(D)$ is just a constant. This interpretation is equivalent to expanding $p(x)$ in powers of $x$ and then replacing $x$ by the operator $D$.

It is interesting that the converse of this proposition is also true and this leads to our first major result.

THEOREM 1. *If $p(x)$ is a polynomial with real coefficients, then $p(D)\mathbf{P} \subset \mathbf{P}$ if and only if $p(x)$ is in **P**.*

To show the only if part, we begin with a polynomial $p(x)$ with real coefficients and assume that $p(D)\mathbf{P} \subset \mathbf{P}$. Let $q_n(x) = p(D)x^n$, $n \geqslant 0$, and $p(x) = \Sigma_0^m a_i x^i$. Then by hypothesis $q_n(x)$ has only real roots. Form the polynomial $r_n(x) = (x^n/n^n)q_n(n/x)$, which also has only real roots. Moreover for $n \geqslant m$

$$q_n(x) = p(D)x^n = \left(\sum_0^m a_i D^i\right)x^n = \sum_0^m a_i n!/(n-i)!x^{n-i},$$

and then

$$\begin{aligned} r_n(x) &= \sum_0^m a_i n!/(n-i)!(x/n)^i \\ &= a_0 + a_1 x + \left\{\sum_2^m a_i[1-1/n][1-2/n]\dots[1-(i-1)/n]x^i\right\} \end{aligned}$$

and we see by taking limits of each term as $n \to \infty$ that $\lim r_n(x) = \Sigma_0^m a_i x^i = p(x)$ for every $x$. Thus $p(x)$ is the limit of polynomials with only real roots. It remains to show that the roots of $p(x)$ are real. Although this will follow from a theorem in complex variables (Hurwitz' theorem), an elementary argument can be used. Let $z_0$ be a root of $p(x)$ with imaginary part $y_0$. Let the roots of $r_n(x)$ be $\gamma_1(n),\ldots,\gamma_m(n)$, all real numbers, and let the leading coefficient of $r_n(x)$ be denoted by $a_m(n)$ whose limit is $a_m$, the leading coefficient of $p(x)$, which is assumed non-zero. Since $|z_0-\gamma_i(n)|\geqslant|y_0|$ for all $i$ because $\gamma_i(n)$ is real, then

$$|r_n(z_0)|=|a_m(n)|\left|\prod_1^m(z_0-\gamma_i(n)\right|\geqslant|a_m(n)|\,|y_0|^m$$

and thus

$$0=|p(z_0)|=\lim|r_n(z_0)|\geqslant\lim|a_m(n)|\,|y_0|^m=|a_m|\,|y_0|^m$$

where all limits are as $n\to\infty$. Hence $y_0$ is zero and $z_0$ is real. The proof is complete.*

We can deduce some results about the location of the roots. Recall that $p(x)$ and $q(x)=(D-\lambda)p(x)$ have the same degree if $\lambda\neq 0$.

LEMMA 1. *Let* $p(x)=\prod_1^n(x-\alpha_i)$ *with* $\alpha_1\leqslant\cdots\leqslant\alpha_n$, *and* $q(x)=(D-\lambda)p(x)$.

a. *If* $\lambda>0$, *the largest root of* $q(x)$ *is in the interval* $(\alpha_n,\alpha_n+(n/\lambda)]$ *and the remaining roots separate the roots of* $p(x)$.

b. *If* $\lambda<0$, *the smallest root of* $q(x)$ *is in* $[\alpha_1+(n/\lambda),\alpha_1)$ *and the remaining roots separate the roots of* $p(x)$.

c. *If* $\lambda=0$, *the roots of* $q(x)$ *separate the roots of* $p(x)$.

Assume $\lambda>0$. The separation of roots is a consequence of an earlier remark (see the proof of proposition 3). Because $e^{-\lambda x}p(x)$ has a root at $\alpha_n$ and at $+\infty$, an application of Rolle's theorem shows that the largest root, $\beta$, of $q(x)=e^{\lambda x}D[e^{-\lambda x}p(x)]$ satisfies $\beta>\alpha_n$. Also since $q(x)=p'(x)-\lambda p(x)$ and $p(\beta)\neq 0$, differentiation leads to

$$\lambda=p'(\beta)/p(\beta)=\sum_{i=1}^n[1/(\beta-\alpha_i)]\leqslant\sum_{i=1}^n[1/(\beta-\alpha_n)]=n/(\beta-\alpha_n)$$

from which it follows that $\beta\leqslant\alpha_n+(n/\lambda)$. This completes the proof.

The case when $\lambda<0$ is proved similarly while that for $\lambda=0$ is a part of proposition 1. That the conclusion is sharp can be seen from the example $q(x)=(D-\lambda)x^n$ whose largest root is $n/\lambda$ when $\lambda>0$.

THEOREM 2. *If* $p(x)$ *is defined by* (2.1), *let* $r(x)=p(D)q(x)$, *where* $q(x)$ *has degree m and* $a,b$ *are the smallest and largest roots respectively of* $q(x)$. *Then the roots of* $r(x)$ *are located in the interval*

$$\left[a+m\sum{}'1/\alpha_i,\ b+m\sum{}''1/\alpha_i\right]$$

*where the single and double primes indicate summation over the negative and positive roots of* $p(x)$, *respectively. The sums are replaced by zero if there are no appropriate roots.*

The proof of this theorem follows quickly by repeated applications of the lemma.

*My thanks to Professor J. H. Smith, Boston College, for this elegant proof.

These results permit the (relatively) easy construction of polynomials with integer coefficients with real and not obvious roots located in desired intervals. Thus the polynomial

$$2x^3 - 12x^2 + 15x - 3 = (-1/2)(D-1)(D-2)^2x^3$$

will have real roots located in [0, 6] by Theorem 2. In fact they are .246679, 1.395310, and 4.358011 correct to six decimal places.

Other applications follow from theorems 1 and 2. The most extensively studied classes of polynomials are probably those that are orthogonal with respect to weight functions on intervals. When the weight function is the constant function one and the interval is finite, then all of these classes are simply related to the *Legendre polynomials* that are orthogonal on $[-1,1]$ with weight function one and can be defined by

$$P_n(x) = 1/(2^n n!)\, D^n(x^2-1)^n.$$

In this case, proposition 1 is applicable to conclude that all roots are real and in $(-1,1)$. The *Laguerre polynomials*, defined by

$$L_n(x) = (1/n!)e^x D^n(e^{-x}x^n) = (1/n!)(D-1)^n x^n$$

form a class orthogonal on the interval $(0,\infty)$ with weight function $e^{-x}$. We deduce from theorems 1 and 2 that all roots of all Laguerre polynomials are real and located in $(0,\infty)$, (zero is not a possible root as can be seen by expanding $L_n(x)$ in powers of $x$). More precisely, theorem 2 establishes the roots of $L_n(x)$ in the interval $(0, n^2]$.

**3. Extending P** We wish to extend the class of operators coming from **P** to a larger class (without changing the domain). The most likely approach is to form the closure of **P** under uniform convergence on all compact sets in the complex plane **C**, which closure we denote by **F**. This means that $f(x)$ is in **F** if there is a sequence from **P** converging uniformly to $f(x)$ on every compact set in **C**. A number of consequences flow from this definition if we use basic complex variable theory.

(a) All functions in **F** are entire, i.e., analytic everywhere in **C**.

(b) Every function $f(x)$ in **F** has only real roots.

(c) If $f(x)$ is in **F** and $\{p_n(x)\}$ is a sequence from **P** converging uniformly to $f(x)$ on every compact set, then $\{p_n^{(k)}(x)\}$ also converges uniformly to $f^{(k)}(x)$ on the same sets for $k = 0, 1, \ldots$ .

By analogy with proposition 4, we would expect the following.

THEOREM 3. *$f(D)\mathbf{P} \subset \mathbf{P}$ for any $f(x)$ in* **F**.

To prove this, let $\{p_n(x)\}$ be the sequence from **P** associated with $f(x)$ and let $q(x)$ be in **P**. By (c) above, $\{p_n^{(k)}(0)\}$ converges to $f^{(k)}(0)$ for all $k$ so that if we write $p_n(x) = \Sigma_0^n a_i(n)x^i$ and $f(x) = \Sigma_0^\infty a_i x^i$, then $\lim a_k(n) = a_k$ as $n \to \infty$ for every $k$. To show that $f(D)q(x)$ is in **P**, form $r_n(x) = p_n(D)q(x) = \Sigma_0^n a_k(n)q^{(k)}(x)$. Then $r_n(x)$ is in **P** by theorem 1. If $m$ is the degree of $q(x)$, then for $n > m$,

$$r_n(x) = \sum_0^m a_k(n)q^{(k)}(x) \to \sum_0^m a_k q^{(k)}(x) = \sum_0^\infty a_k q^{(k)}(x) = f(D)q(x).$$

The limit is as $n \to \infty$ and we have used, in taking this limit, the fact that each sum contains at most a finite number of non-zero terms because of the degree of $q(x)$. By a result proved above (proof of theorem 1), the polynomial $f(D)q(x)$ is in **P** and this proves the theorem.

It is now of interest to determine constructively the exact form of the functions in **F**. From elementary calculus, $e^{bx} = \lim[1+(bx/n)]^n$ as $n \to \infty$. Thus $e^{bx}$, $b$ real, being the limit of polynomials in **P**, is in **F** if the convergence is uniform. To see this, write

$$[1+(bx)/n]^n = \sum_0^n \binom{n}{k}(bx/n)^k = \sum_0^\infty \gamma_k(n)(bx)^k/k!,$$

with $\gamma_k(n) = n!/(n-k)!n^k = [1-1/n][1-2/n]\dots[1-(k-1)/n]$ for $2 \leqslant k \leqslant n$, $\gamma_k(n) = 1$ for $k = 0, 1$ and $\gamma_k(n) = 0$ if $k > n$. For a positive integer $N$,

$$\begin{aligned}\left|e^{bx} - [1+(bx/n)]^n\right| &\leqslant \sum_{k=1}^N [1-\gamma_k(n)]|bx|^k/k! + \sum_{k=N+1}^\infty |bx|^k/k! \\ &\leqslant \sum_{k=1}^N [1-\gamma_k(n)]|bR|^k/k! + \sum_{k=N+1}^\infty |bR|^k/k!\end{aligned}$$

for $|x| \leqslant R$, $R > 0$ and $x$ in **C**. We have used $0 \leqslant \gamma_k(n) \leqslant 1$ for all $k, n$. Given $\varepsilon > 0$, the integer $N$ can be chosen so that the second sum is less than $\varepsilon/2$ because of the convergence of the series. The first sum can be made less than $\varepsilon/2$ because $\gamma_k(n) \to 1$ as $n \to \infty$ for each $k$, $1 \leqslant k \leqslant N$, fixed $N$. Thus the convergence is uniform for $|x| \leqslant R$ and consequently $e^{bx}$ is in **F** for any $b$ real. Similarly $e^{-cx^2}$ is in **F** because it is the limit of $[1-(cx^2/n)]^n$, uniform on compact sets as before, provided $c > 0$, for only then are the polynomials in **P**. Of course $c = 0$ creates no difficulty since **F** contains the constant functions. Now let $\{\alpha_k\}$ be a sequence of non-zero real numbers, and consider the function

$$f(x) = \prod_1^\infty [1-(x/\alpha_k)]e^{x/\alpha_k}, \qquad \sum_1^\infty 1/\alpha_k^2 < \infty. \tag{3.1}$$

To show that $f(x)$ is in **F**, we must show the existence of a sequence from **P** converging uniformly to $f(x)$ on every compact set in **C**. To this end, let

$$f_m(x) = \prod_1^m [1-(x/\alpha_k)]e^{x/\alpha_k} = e^{\gamma_m x}\prod_1^m [1-(x/\alpha_k)],$$

where $\gamma_m = \Sigma_1^m 1/\alpha_k$ and observe that $f_m(x)$ is in **F** because **P**, and hence **F**, is closed under finite products. Then

$$|f(x) - p_n(x)| \leqslant |f(x) - f_m(x)| + |f_m(x) - p_n(x)|. \tag{3.2}$$

For each $n$, let $K_n = \{x \text{ in } \mathbf{C} : |x| \leqslant n\}$. Because the Weierstrass factor theorem asserts the uniform convergence of the infinite product of (3.1) to $f(x)$ on any compact set in **C**, in particular on $K_n$, we can choose $m$ so that the first term in (3.2) is less than $1/(2n)$ for $x$ in $K_n$. Then with $m$ fixed, $p_n(x)$ is chosen from **P** to make the second term less than $1/(2n)$ for the same $x$. Thus $|f(x) - p_n(x)| < 1/n$ on $K_n$. Here $p_n(x)$ is not necessarily of degree $n$. Since $K_n \subset K_{n+1}$ and any compact set is contained in some $K_n$, the result is a sequence from **P** converging uniformly to $f(x)$ on any compact set. Thus $f(x)$ is in **F** and we can conclude that **F** contains all functions of the form

$$Ax^\lambda e^{bx-cx^2}\prod_1^\infty [1-(x/\alpha_k)]e^{x/\alpha_k}, \tag{3.3}$$

with $A, b$ real, $c \geqslant 0$, $\lambda$ a non-negative integer and $\Sigma_1^\infty 1/\alpha_k^2 < \infty$. This is the famous *Pólya-Schur* class of entire functions. Note that **P** is included in this class by letting $\alpha_k = \infty$ for $k$ beyond some value and choosing the remaining constants suitably.

It is very interesting that **F** contains no more than these functions. This is technically more difficult to prove and will not be shown although it will be used in what follows.

THEOREM 4. *If $f(x)$ is entire, then $f(D)\mathbf{P} \subset \mathbf{P}$ if and only if $f(x)$ is in the Pólya-Schur class.*

The sufficiency follows from theorem 3. To show the necessity, write $f(x) = \Sigma_0^\infty a_k x^k$. Then, by hypothesis, the polynomial $p_n(x) = f(D)x^n = \Sigma_0^n a_k [n!/(n-k)!] x^{n-k}$ has only real roots and consequently so has $q_n(x) = (x/n)^n p_n(n/x) = \Sigma_0^n a_k \gamma_k(n) x^k$ where $\gamma_k(n) = 1$ for $k = 0, 1$ and $\gamma_k(n) = [1 - 1/n][1 - 2/n]\dots[1 - (k-1)/n]$ for $k \geqslant 2$. An argument almost exactly the same as the one used above to show that $e^{bx}$ is in **F** can be applied here to show that $\{p_n(x)\}$ converges uniformly to $f(x)$ on compact sets. Thus $f(x)$ is in **F**, the Pólya-Schur class.

It is unnecessary to assume $f(x)$ entire, just analyticity at $x = 0$ will do, although our proof would have to be modified to show this. This result is used below.

A good example of theorem 4 is the *Hermite* polynomial $H_n(x)$ that, while normally defined by $H_n(x) = (-1)^n e^{x^2} D^n e^{-x^2}$, can be defined by

$$H_n(x/2) = e^{-D^2} x^n. \tag{3.4}$$

The simplest interpretation for $e^{-D^2}$ comes from using the Maclaurin series for the exponential function and replacing the variable by $-D^2$. Using (3.4), we conclude that $H_n(x)$ has only real roots since $e^{-x^2}$ is in **F**.

The class of Hermite polynomials is an example of the *Appell* classes of polynomials defined by

$$\sum_0^\infty A_n(x) z^n/n! = A(z) e^{xz} \tag{3.5}$$

with $A(z)$ analytic at $z = 0$ and $A(0) \neq 0$. If we write $A(z) = \Sigma_0^\infty a_k z^k$, then by multiplying this series for $A(z)$ with the power series (powers of $x$) of $e^{xz}$ and equating coefficients of like powers of $z$ in (3.5), we get $A_n(x) = n! \Sigma_0^n a_k x^{n-k}/(n-k)!$. This can be written as

$$A_n(x) = \sum_0^n a_k D^k x^n = \sum_0^\infty a_k D^k x^n = A(D) x^n. \tag{3.6}$$

Then Theorem 4 guarantees that $A_n(x)$ has only real roots if $A(x)$ is in **F**. The converse follows from the *proof* of the necessity of theorem 4 since only the action of the operator on the polynomials $x^n$ (here indicated by $A(D)$ in (3.6)) was used in the proof. Also used is the extension of theorem 4 mentioned after the proof. We have shown the following.

THEOREM 5. *The Appell polynomials defined by* (3.5) *have only real roots if and only if $A(x)$ is in the Pólya-Schur class* (*with* $\lambda = 0$).

The restriction $\lambda = 0$ arises because $A(0) \neq 0$ for Appell polynomials. $A(x) = e^{-cx}$ leads to $A_n(x) = (x - c)^n$ so that expansions in Appell polynomials can be considered generalizations of Taylor series. In fact, they form the basis of polynomial expansions in the well-known Bourbaki text on real variables [2].

**4. References and notes** In this section we will trace through the source material.

Proposition 1 (and thus proposition 2) is of course well known. Proposition 3 is a consequence of a result attributable to Poulain [7] that was in response to a "question" of Hermite who very likely had the solution. It can also be found as a problem in the text by Hardy, [3; p. 277, prob. 27].

The estimate for the roots of the Laguerre polynomials at the end of section 2 is not sharp since the largest root of $L_n(x)$ is known to be of the order $4n$, [8, 124]. The properties from complex variables quoted in section 3 are contained in a standard course. Theorem 3 follows from a result of Pólya, [5, section 1]. Theorem 5 has appeared in [1].

In addition to Pólya and Schur [6] (who refer to this class as the class of *entire functions of type II*), the Polya-Schur class has been studied by many others; see Hirschman and Widder [4] and the references there. The name of Laguerre is often associated with this class since he was the first to show that it is the uniform limit of sequences of polynomials with only real roots.

The definition of Hermite polynomials adopted in (3.4) can be found in [4, 178] although it is not difficult to verify directly. It might reasonably be expected that all of the standard orthogonal polynomials would fit under our theory as the examples of the Legendre, Laguerre, and Hermite polynomials suggest. This is not so because many formulas of the Rodrigues' type, which can be used to define many orthogonal polynomials, involve differential operators acting on non-polynomial (and non-entire) functions, and this would involve a further extension of the work of this paper from the class **P** to a class not contained in **F**. In general, this should not present any serious theoretical difficulty.

It is a reasonable extension of theorem 3 that $f(D)\mathbf{F} \subset \mathbf{F}$ should be true for every $f(x)$ in **F**. This is not quite correct since the series involved in the interpretation of the expression $f(D)g(x)$ ($g(x)$ in **F**) may diverge. Under certain conditions $f(D)g(x)$ is in **F** [5, 242].

A companion problem to this work is to conclude something about the roots of $q(x)$ from a knowledge of the roots of $r(x)$ in the expression $p(D)q(x) = r(x)$, in effect the reverse of our approach. We can now write $q(x) = [1/p(D)]r(x)$ and consider $q(x)$ as the *transform* of $r(x)$. Because real roots of a function are closely related to the changes in sign of the function, this problem has given rise to an extensive theory of *variation diminishing transforms*. See Pólya, [5, section 2], for early results and Hirschman and Widder, [4], for a complete theory.

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# On Some Symmetric Sets of Unit Vectors

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In this note we start with a given set of symmetric unit vectors $\mathbf{A}_1, \mathbf{A}_2, \dots, \mathbf{A}_n$ in a Euclidean space with $\Sigma \mathbf{A}_k = \mathbf{0}$ and consider conditions on real numbers $x_1, x_2, \dots, x_n$ that allow us to conclude that

$$\Sigma x_k \mathbf{A}_k = \mathbf{0} \Rightarrow x_1 = x_2 = \cdots = x_n.$$

Different kinds of "symmetry" will lead to different conclusions.

First we consider a known result concerning symmetric unit length complex numbers in the plane. One natural such set is the $n$th roots of unity

$$1, \omega, \omega^2, \dots, \omega^{n-1}$$

where $\omega = e^{2\pi i/n}$. Since $\omega^n - 1 = 0$, it follows immediately that

$$1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0.$$

A converse result also holds, for if

$$x_0 + x_1\omega + x_2\omega^2 + \cdots + x_{n-1}\omega^{n-1} = 0, \tag{1}$$

where $x_0 \geqslant x_1 \geqslant x_2 \geqslant \cdots \geqslant x_{n-1}$, then $x_0 = x_1 = x_2 = \cdots = x_{n-1}$.

Our proof is indirect, so we assume the $x_i$'s are not all equal. On multiplying (1) by $1 - \omega$, we obtain

$$x_0 - x_{n-1} = (x_0 - x_1)\omega + (x_1 - x_2)\omega^2 + \cdots + (x_{n-2} - x_{n-1})\omega^{n-1}. \tag{2}$$

First we consider the case

$$x_0 = x_1 = \cdots = x_k > x_{k+1} = x_{k+2} = \cdots = x_{n-1}.$$

Here (2) reduces to $x_0 - x_{n-1} = (x_k - x_{k+1})\omega^{k+1} = (x_0 - x_{n-1})\omega^{k+1}$, from which $x_0 = x_{n-1}$. For all other cases, we apply the triangle inequality to

$$|x_0 - x_{n-1}| = |(x_0 - x_1)\omega + (x_1 - x_2)\omega^2 + \cdots + (x_{n-2} - x_{n-1})\omega^{n-1}|,$$

which yields

$$x_0 - x_{n-1} < (x_0 - x_1) + (x_1 - x_2) + \cdots + (x_{n-2} - x_{n-1}) = x_0 - x_{n-1},$$

giving the desired contradiction.

For an extension of the above result, we now consider a set of $n + 1$ distinct concurrent unit vectors $\mathbf{A}_0, \mathbf{A}_1, \dots, \mathbf{A}_n$ in $E^n$ that are equally inclined to each other. It is geometrically intuitive that

$$\mathbf{A}_0 + \mathbf{A}_1 + \cdots + \mathbf{A}_n = 0 \tag{2}$$

and for which one can give a number of different proofs. First, as suggested by one of the referees, is to use mathematical induction on the dimension $n$. Secondly, we can use barycentric coordinates noting that from the centroid

$$\mathbf{G} = (\mathbf{A}_0 + \mathbf{A}_1 + \cdots + \mathbf{A}_n)/(n+1),$$

the volumes spanned by any $n$ of the endpoints of the vectors $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_n$ are all equal and the same is true from the origin of the vectors. Thirdly, as suggested by my colleague Jim Pounder, the centroid is unique and if it did not coincide with the origin, then by applying the group of transformations that take the given set of vectors into itself, we would generate many different centroids. Finally, we give an easy self-contained analytic proof based on linear independence. Since the given space is of dimension $n$, there exist sets of $n$ linearly independent vectors. Any set of $n$ of the vectors $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_n$ are linearly independent since the volumes spanned by each set are equal and $> 0$. Hence, there is a representation

$$\mathbf{A}_0 = x_1\mathbf{A}_1 + x_2\mathbf{A}_2 + \cdots + x_n\mathbf{A}_n. \tag{3}$$

We now take the scalar product of (3) with each of $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_n$, giving

$$1 = kS, \tag{4}$$

$$k = x_i + (S - x_i)k, \qquad i = 1, 2, \ldots, n \tag{5}$$

where $k = \mathbf{A}_i \cdot \mathbf{A}_j$ $(i \neq j)$ and $S = \sum x_i$. Summing (5) over $i$, gives

$$nk = S + nkS - kS.$$

Now replacing $S$ by $1/k$, we get $(k-1)(nk+1) = 0$. Since the $\mathbf{A}_i$'s are distinct, $k \neq 1$ so that $k = -1/n$. We now calculate the square of the length of the sum of the $n+1$ vectors, i.e.,

$$(\mathbf{A}_0 + \mathbf{A}_1 + \cdots + \mathbf{A}_n)^2 = n + 1 + 2k\binom{n+1}{2} = 0,$$

which gives the desired result.

It now follows as in our introductory example that if

$$x_0\mathbf{A}_0 + x_1\mathbf{A}_1 + \cdots + x_n\mathbf{A}_n = 0, \tag{6}$$

where here the $x_i$'s are any real numbers, not necessarily monotonic (since we have more symmetry here), then the $x_i$'s are all equal. Just square (5) to give

$$\sum_i x_i^2 - 2\sum_{i<j} x_i x_j/n = 0 = \sum_{i<j} (x_i - x_j)^2/n.$$

It is worth noting that one cannot have a set of more than $n+1$ distinct unit vectors in $E^n$ that are equally inclined to each other. If there were $n+2$ such vectors, then as before the sum of every subset of $n+1$ of the vectors must be zero. This implies that the vectors are not distinct, which contradicts the hypotheses.

For our final example, we leave an open problem. Consider a hypercube in $E^n$ with vertices $(\pm 1, \pm 1, \ldots, \pm 1)$. The set of the $2^n$ vectors from the origin to the vertices is a symmetric set of vectors with sum zero. It is possible to order this set of vectors $\mathbf{A}_i$ such that if

$$x_1\mathbf{A}_1 + x_2\mathbf{A}_2 + \cdots + x_{2n}\mathbf{A}_{2n} = 0,$$

where $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_{2n}$, then all the $x_i$'s must be equal. The open problem is to find all such possible orderings. In particular, it is immediate that some of the orderings of a Grey Code arrangement are possible, i.e., the number of sign changes in the components between two adjacent vectors including the last with the first is exactly one. For example in $E^3$, we have the ordering

$$(1,1,1),(1,-1,1),(-1,-1,1),(-1,1,1),(-1,1,-1),$$
$$(-1,-1,-1),(1,-1,-1),(1,1,-1).$$

This ordering works since the third components are in monotonic order. One can also show that the cyclic permutations of these 8 vectors are all possible orderings. Is this also true in $E^n$?

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# A Coordinate Approach to the AM-GM Inequality

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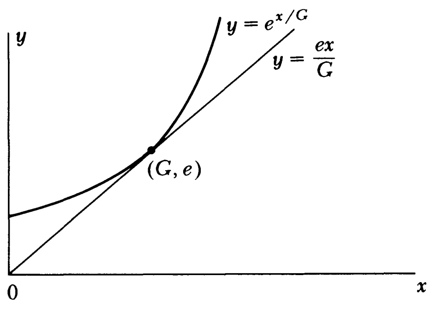
Let $a_1, a_2, \ldots, a_n$ be $n$ positive numbers with arithmetic mean $A$ and geometric mean $G$. The *AM–GM* Inequality states that $A \geqslant G$ with equality if and only if $a_1 = a_2 = \cdots = a_n$.

The graph of $y = e^{x/G}$ is concave upward and thus the tangent line $y = (ex/G)$ at $(G, e)$ lies below the curve. To show $A \geqslant G$, we substitute $x = a_i$ $(i = 1,2,\ldots,n)$ successively into $e^{x/G} \geqslant (ex/G)$ and multiply. Hence

$$e^{(a_1+a_2+\cdots+a_n)/G} \geqslant \left(\frac{ea_1}{G}\right)\left(\frac{ea_2}{G}\right)\cdots\left(\frac{ea_n}{G}\right) = e^n.$$

Thus, we have $nA/G \geqslant n$ or $A \geqslant G$, with equality if and only if $a_1 = a_2 = \cdots = a_n = G$.




**Acknowledgement.** This proof is a geometric variation of the proof suggested by G. L. Alexanderson to Ivan Niven in *Maxima and Minima Without Calculus*, MAA, 1981, pp. 240–241.

orderings of a Grey Code arrangement are possible, i.e., the number of sign changes in the components between two adjacent vectors including the last with the first is exactly one. For example in $E^3$, we have the ordering

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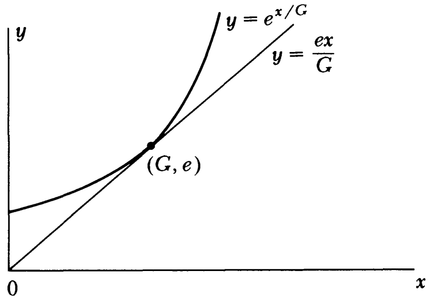
Let $a_1, a_2, \ldots, a_n$ be $n$ positive numbers with arithmetic mean $A$ and geometric mean $G$. The *AM–GM* Inequality states that $A \geqslant G$ with equality if and only if $a_1 = a_2 = \cdots = a_n$.

The graph of $y = e^{x/G}$ is concave upward and thus the tangent line $y = (ex/G)$ at $(G, e)$ lies below the curve. To show $A \geqslant G$, we substitute $x = a_i$ $(i = 1, 2, \ldots, n)$ successively into $e^{x/G} \geqslant (ex/G)$ and multiply. Hence

$$e^{(a_1+a_2+\cdots+a_n)/G} \geqslant \left(\frac{ea_1}{G}\right)\left(\frac{ea_2}{G}\right)\cdots\left(\frac{ea_n}{G}\right) = e^n.$$

Thus, we have $nA/G \geqslant n$ or $A \geqslant G$, with equality if and only if $a_1 = a_2 = \cdots = a_n = G$.




**Acknowledgement.** This proof is a geometric variation of the proof suggested by G. L. Alexanderson to Ivan Niven in *Maxima and Minima Without Calculus*, MAA, 1981, pp. 240–241.

# PROBLEMS

LOREN C. LARSON, *editor*
St. Olaf College

GEORGE GILBERT, *associate editor*
Texas Christian University

## Proposals

*To be considered for publication, solutions should be received by March 1, 1992.*

**1378.** *Proposed by Stephen G. Penrice, Arizona State University, Tempe, Arizona.*

Let $(i)_j$ denote the falling product $i(i-1)\cdots(i-j+1)$ and let $(i)_0=1$. Show that for all positive integers $n$ and $k$

$$\frac{(n+k)_{k+1}}{2(k!)^2}\sum_{i=1}^{k}(k)_{i-1}(n+k-i)_{k-i}$$

is a triangular number.

**1379.** *Proposed by John O. Kiltinen, Northern Michigan University, Marquette, Michigan.*

Call an operation $*$ on a nonempty set $A$ *self-seeking* if every permutation of $A$ is an automorphism from $(A, *)$ to $(A, *)$. Such operations have no isomorphic copies on the set other than themselves. Describe all the self-seeking operations, if any, on $A$.

**1380.** *Proposed by Thoddi C. T. Kotiah, Utica College of Syracuse University, Utica, New York.*

Let $P(x)=x^n-a_1x^{n-1}-a_2x^{n-2}-\cdots-a_n$, where the $a_i$ are nonnegative real numbers, not all zero. Let $s=\sum_{i=1}^n a_i$ and $v=\sum_{i=1}^n ia_i$. Prove that a lower bound for the positive real zero of $P(x)$ is $s^{s/v}$.

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ASSISTANT EDITORS: Clifton Corzat, Bruce Hanson, Richard Kleber, Kay Smith, and Theodore Vessey, *St. Olaf College and* Mark Krusemeyer, *Carleton College. We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals should be accompanied by solutions, if at all possible, and by any other information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. An asterisk (*) next to a problem number indicates that neither the proposer nor the editors supplied a solution.*

*Solutions should be written in a style appropriate for* Mathematics Magazine. *Each solution should begin on a separate sheet containing the solver's name and full address.*

*Solutions and new proposals should be mailed in duplicate to Loren Larson, Department of Mathematics, St. Olaf College, 1520 St. Olaf Ave., Northfield, MN 55057-1098 or mailed electronically via fax: (507) 663-3549 or e-mail: larson@stolaf.edu.*

**1381.** *Proposed by Mihály Bencze, Braşov, Romania.*

Find all integer solutions of the following ($n$ and $k$ are positive integers).

a. $(x+y)^{2n+1}=x^{2n}+y^{2n}$

b*. $(x_1+x_2+\cdots+x_k)^{kn+k-1}=x_1^{kn}+x_2^{kn}+\cdots+x_k^{kn}$

**1382.** *Proposed by Michael Golomb, Purdue University, West Lafayette, Indiana.*

Suppose $f$ is a real-valued function of class $C^\infty$ near $x_0 \in \mathbf{R}$, and $g$ is a real-valued function of class $C^\infty$ near $f(x_0)$. Prove that if $g \circ f - e$ ($e$ the identity function) has a zero of order $n$ $(1 \leqslant n \leqslant \infty)$ at $x_0$, then $f \circ g - e$ has a zero of the same order at $f(x_0)$.

# Quickies

*Answers to the Quickies are on page 281.*

**Q781.** *Proposed by Stanley Rabinowitz, Westford, Massachusetts.*

Let $P$ be a point inside triangle $ABC$. Let $X$, $Y$, and $Z$ be the centroids of triangles $BPC$, $CPA$, and $APB$ respectively. Prove that segments $AX$, $BY$, and $CZ$ are concurrent.

**Q782.** *Proposed by Hugh Noland, Chico, California.*

Let $f$ be a real function. Show that if $f'$ is defined at $c$, and if $\lim_{x\to c} f'(x)$ exists, then $f'$ is continuous at $c$.

**Q783.** *Proposed by M. S. Klamkin and A. Liu, University of Alberta, Edmonton, Canada.*

If all the vertices of a regular $n$-gon are lattice points in a plane tessellated by equilateral triangles, then $n=3$ or $n=6$.

# Solutions

**Application of AM-GM inequality** **October 1990**

**1353.** *Proposed by Mihály Bencze, Braşov, Romania.*

Prove that

$$(\sqrt{2}-1)\left(\sqrt[3]{6}-\sqrt{2}\right)\cdots\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)<\frac{n!}{(n+1)^n}.$$

I. *Solution by Joe Howard, New Mexico Highlands University, Las Vegas, New Mexico.*

By the arithmetic mean-geometric mean inequality,

$$1\cdot(\sqrt{2}-1)\cdot\left(\sqrt[3]{6}-\sqrt{2}\right)\cdots\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)$$

$$< \left( \frac{1 + (\sqrt{2} - 1) + \left(\sqrt[3]{6} - \sqrt{2}\right) + \cdots + \left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}\right)}{n+1} \right)^{n+1}$$

$$= \frac{(n+1)!}{(n+1)^{n+1}} = \frac{n!}{(n+1)^n}.$$

II. *Solution by Nick Lord, Tonbridge School, Kent, England.*
Let $u_k = \sqrt[k+1]{(k+1)!} - \sqrt[k]{k!}$. We show the slightly stronger inequality:

$$u_k < \frac{1}{(1+1/k)^k} = \frac{k \cdot k^{k-1}}{(k+1)^k}.$$

To see this, apply the arithmetic mean-geometric mean inequality to

$$\underbrace{\frac{\sqrt[k]{k!}}{k}, \ldots, \frac{\sqrt[k]{k!}}{k}}_{k \text{ times}}, \frac{1}{(1+1/k)^k}.$$

This yields

$$\frac{1}{k+1}\left[k\left(\frac{\sqrt[k]{k!}}{k}\right) + \frac{1}{(1+1/k)^k}\right] > \left(\frac{1}{k^k}k!\frac{1}{(1+1/k)^k}\right)^{1/(k+1)}$$

$$= \left(\frac{(k+1)!}{(k+1)^{k+1}}\right)^{1/(k+1)}$$

$$= \frac{(k+1)!^{1/(k+1)}}{k+1}$$

and the result follows.

Now, using this, we obtain

$$u_1 u_2 \ldots u_n < (1 \cdot 2 \ldots n)\frac{1^0}{2^1} \cdot \frac{2^1}{3^2} \cdots \frac{n^{n-1}}{(n+1)^n} = \frac{n!}{(n+1)^n}$$

as required.

*Also solved by David Callan, E. Braüne (Austria), Jiro Fukuta (Japan), A. A. Jagers (The Netherlands), Julie Kerr (student), Y. H. Harris Kwong, Kee-Wai Lau (Hong Kong), Richard Pfiefer, Leland Prior, Pasquale Saeva, Rene L. Schilling (Germany), John S. Sumner, Edward T. H. Wang (Canada), and the proposer.*

Braüne *proved the somewhat more general inequality: Let $p_j$ and $r_j$ be positive integers with $p_2 = 2 < p_3 < \cdots < p_{n+1}$, and $r_2 = 2 < r_3 < \cdots < r_{n+1}$, where $r_{j+1} = r_j + K$, ($j = 2, 3, \ldots, n$, $K \geqslant 1$, an integer). Also, suppose that $p_{j+1}^{r_j} > (p_2 p_3 \cdots p_j)^K$, $j = 2, 3, \ldots, n$. Then*

$$\left((p_2)^{\frac{1}{r_2}} - 1\right)\left((p_2 p_3)^{\frac{1}{r_3}} - (p_2)^{\frac{1}{r_2}}\right)\cdots\left((p_2 \cdots p_{n+1})^{\frac{1}{r_{n+1}}} - (p_2 \cdots p_n)^{\frac{1}{r_n}}\right) < \frac{(p_2 p_3 \cdots p_{n+1})^{\frac{n+1}{r_{n+1}}}}{(n+1)^{n+1}}.$$

*We get the result of the problem by setting $K = 1$, $p_j = r_j = j$, for $j = 2, \ldots, n+1$. The condition is satisfied because $j! \leqslant 2j^{j-2} < (j+1)^j$ for all $j \geqslant 2$.*

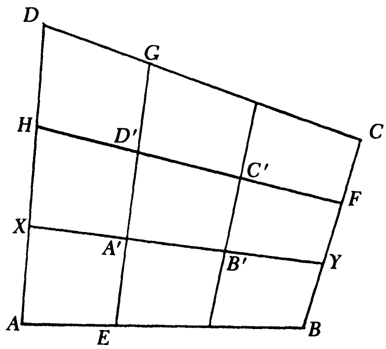
**Quadrilateral subdivision** **October 1990**

**1354.** *Proposed by Frank Schmidt, Bryn Mawr College, Bryn Mawr, Pennsylvania, and Rodica Simion, George Washington University, Washington, DC.*

Let $ABCD$ be a convex quadrilateral in the plane with trisection points joined as in the figure to form nine smaller quadrilaterals.

a. Show that the area of $A'B'C'D'$ is one-ninth the area of $ABCD$.

b. Give necessary and sufficient conditions so that all nine quadrilaterals have equal area.




I. *Solution by Václav Konečný, Ferris State University, Big Rapids, Michigan.*

a. First we show that the points $A', B', C', D'$ are also trisection points. Naming some of the trisection points along the sides of $ABCD$ as shown in the figure, we have $\overrightarrow{EF} = \frac{2}{3}\overrightarrow{AC}$ and $\overrightarrow{HG} = \frac{1}{3}\overrightarrow{AC}$. Therefore, triangles $D'GH$ and $D'EF$ are similar, $|\overrightarrow{D'F}| = 2|\overrightarrow{D'H}|$, and $D'$ is a trisection point. By the symmetry of the problem, so are $A'$, $B'$, and $C'$.

It follows that $\overrightarrow{A'C'}[=\frac{1}{2}\overrightarrow{EF}] = \frac{1}{3}\overrightarrow{AC}$ and $\overrightarrow{B'D'} = \frac{1}{3}\overrightarrow{BD}$. Therefore, if $\theta$ is the angle between $\overrightarrow{AC}$ and $\overrightarrow{BD}$, we have

$$\begin{aligned}\text{Area of } A'B'C'D' &= \tfrac{1}{2}|\overrightarrow{A'C'}|\,|\overrightarrow{B'D'}|\sin\theta\\ &= \tfrac{1}{2}\cdot\tfrac{1}{9}|\overrightarrow{AC}|\,|\overrightarrow{BD}|\sin\theta\\ &= \tfrac{1}{9}\text{Area of } ABCD.\end{aligned}$$

b. A necessary and (clearly) sufficient condition is that $ABCD$ be a parallelogram. To show necessity, note that if $AEA'X$ and $XA'D'H$ have equal area, then since triangles $EA'X$ and $D'A'X$ have equal area, so do the "leftover" triangles $EAX$ and $D'HX$. But then $AD$ must be parallel to $ED'$, that is, to $EG$; by the symmetry of the problem, we see that if all nine quadrilaterals have equal area, they are all parallelograms.

II. *Comment by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

Results generalizing part a, and easily implying part b, are given in Donald Batman and Murray Klamkin, Solution to Problem E2423 (1974, *Amer. Math. Monthly*, pp. 666–668). Part a also appeared previously as Problem E1548 (1962, *Amer. Math. Monthly*).

*Also solved by J. C. Binz (Switzerland), David Callan, Chico Problem Group, Jordi Dou (Spain; two solutions), Frank Eccles, Mordechai Falkowitz (Israel), Jiro Fukuta (Japan), Stewart Gleason, John F. Goehl, Jr., Michael Golomb, E. C. Greenspan and S. A. Greenspan, H. Guggenheimer, Michael E. Hardy (student), Francis M. Henderson, R. Daniel Hurwitz, Julie Kerr (student), Alexander Khoury, Václav Konečný (second solution), Sidney Kung (part a), Lamar University Problem Group, Nick Lord (England), Metropolitan M.A.A. Student Chapter of New York City, Richard E. Pfiefer, Ioan Sadoveanu, Jyotirmoy Sarkar, John S.Sumner and Kevin L. Dove, R. S. Tiberio, Robert L. Young, and the proposers.*

*Many solutions used some combination of coordinatization and vector analysis.* Binz, Dou, Falkowitz, Khoury, Konečný, Kung, Sumner *and* Dove, *and* Tiberio *provided generalizations to "finer" subdivisions than trisections, all of which follow from the work of* Batman *and* Klamkin *cited in Klamkin's comment.* Falkowitz, Pfiefer, Sadoveanu, *and* Sarkar *generalized from trisection to subdivision in the ratio* $\lambda : 1 - 2\lambda : \lambda$. Golomb *pointed out that the area of XYFH is one-third of the area of ABCD.* Kerr *mentioned that part a had been used at an Olympiad training session.* Sadoveanu *showed that if ABCD is taken to be a quadrilateral in 3-space whose "opposite" sides are subdivided in corresponding ratio (but not necessarily trisected), then the intersection points A', B', C', D' will still exist, and he gave a formula for the ratio of the volumes of the tetrahedra A'B'C'D', ABCD in this case.* Callan, Metropolitan MAA Student Chapter of New York City, *and* Tiberio *provided yet another reference for part a*: B. Greenberg, "*That Area Problem*" *Mathematics Teacher*, 64 (1971), *pp.* 79–80.

**Extrema on a hypercube** **October 1990**

**1355.** *Proposed by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.*

Determine the extreme values of

$$F \equiv x_1 x_2 \cdots x_n - (x_1 + x_2 + \cdots + x_n),$$

where $b \geqslant x_i \geqslant a \geqslant 0$ for all $i$.

*Solution by Christos Athanasiadis, student, Massachusetts Institute of Technology, Cambridge, Massachusetts.*

We note that as a function of $x_j$ alone, $F$ is linear, so it attains its extreme values at $x_j = a$ and $x_j = b$. It easily follows that the extreme values of $F$ are attained at $n$-tuples $(x_1, x_2, \dots, x_n)$ for which $x_i \in \{a, b\}$ for all $i$. Thus, by the symmetry of $F$, the extreme values of $F$ are among the numbers $c_0, c_1, \dots, c_n$ defined by

$$c_k = a^{n-k} b^k - (n-k)a - kb.$$

Note that $c_{k+1} - c_k = (a^{n-k-1} b^k - 1)(b - a)$ for $k = 0, 1, \dots, n-1$. We now distinguish three cases.

*Case 1.* $a \geqslant 1$. Then $c_{k+1} \geqslant c_k$ for all $k$, so the minimum of $F$ is $c_0 = a^n - na$ and the maximum is $c_n = b^n - nb$.

*Case 2.* $b \leqslant 1$. Then $c_{k+1} \leqslant c_k$ for all $k$, so the minimum is $c_n$ and the maximum is $c_0$.

*Case 3.* $a < 1 < b$. In this case, $c_{k+1} \geqslant c_k$ if and only if

$$k \geqslant \frac{(n-1)\log(1/a)}{\log(b/a)}. \qquad (*)$$

Thus, if $k$ is the smallest integer for which $(*)$ holds, then the minimum of $F$ is $c_k$, while the maximum is $\max\{c_0, c_n\}$.

*Also solved by Robert A. Agnew, Seung-Jin Bang (Korea), Russell Jay Hendel, A. A. Jagers (The Netherlands), Kee-Wai Lau (Hong Kong), Heinz-Jürgen Seiffert (Germany), John S. Sumner and Kevin L. Dove and Tom McDonald, and the proposer. We also received one incomplete solution.*

**Collinearity and symmetry** **October 1990**

**1356.** *Proposed by Hüseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let $P, Q$ be points taken on the side $BC$ of a triangle $ABC$, in the order $B, P, Q, C$. Let the circumcircles of $PAB, QAC$ intersect at $M(\neq A)$ and those of $PAC, QAB$ at $N$. Show that $A, M, N$ are collinear if and only if $P$ and $Q$ are symmetric in the midpoint $A'$ of $BC$.

*Solution by Christos Athanasiadis, student, Massachusetts Institute of Technology, Cambridge, Massachusetts.*

Let $K$ and $L$ be the points of intersection of the line $BC$ with the lines $AM$ and $AN$ respectively. Suppose that the line $BC$ is the $x$-axis of a coordinate system with origin $B$, and let $a$, $p$, $q$, $k$, and $l$ denote the $x$-coordinates of $C$, $P$, $Q$, $K$, and $L$ respectively. The point $K$ is on the radical axis of the circumcircles of $PAB$ and $QAC$, hence its powers $k(k-p)$ and $(k-q)(k-a)$ with respect to these two circles are equal. It follows that $k = aq/(a+q-p)$. Similarly we have $l = ap/(a+p-q)$, interchanging the roles of $p$ and $q$. We easily find that $l = k$ if and only if $p+q=a$ and the result follows.

*Also solved by Raúl Marin Carrera (student, Mexico), Jordi Dou (Spain), Jiro Fukuta (Japan), Václav Konečný, Alvaro Avila Márquez (student, Mexico), Richard E. Pfiefer, Ioan Sadoveanu, Jyotirmoy Sarkar (student), Seshadri Sivakumar (Canada), John S. Sumner, and the proposer.*

Pfiefer *obtained the solution by inverting the figure through a circle with center A.*

**Characteristic polynomials of tridiagonal matrix** **October 1990**

**1357.** *Proposed by George Gilbert, Texas Christian University, Fort Worth, Texas.*

Let $\mathbf{A}_n = (a_{ij})$ be the $n \times n$ (band) matrix with

$$a_{ij} = \begin{cases} 1, & \text{if } |i-j| \leqslant 1; \\ 0, & \text{otherwise.} \end{cases}$$

Define $\mathbf{B}_n$ just as $\mathbf{A}_n$ except that $b_{nn} = 2$, and $\mathbf{C}_n$ just as $\mathbf{A}_n$ except $c_{n,n-1} = 2$.

a. Prove that the characteristic polynomial of $\mathbf{B}_n$ divides that of $\mathbf{A}_{2n}$.

b. Prove that the characteristic polynomial of $\mathbf{C}_n$ divides that of $\mathbf{A}_{2n-1}$.

c. Prove that $\mathbf{C}_n$ is diagonalizable.

I. *Solution by David Callan, University of Wisconsin, Madison, Wisconsin.*

Define a fourth matrix $\mathbf{D}_n$ just as $\mathbf{A}_n$ except $d_{nn} = 0$. We show that $\mathbf{A}_{2n-1}$ and $\mathbf{A}_{2n}$ are similar respectively, to

$$\begin{pmatrix} \mathbf{A}_{n-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_n \end{pmatrix} \text{ and } \begin{pmatrix} \mathbf{D}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_n \end{pmatrix}.$$

To this end, define a nonsingular $n \times n$ matrix $\mathbf{P}_n$ as follows, for $n$ even and $n$ odd (shown here for $n = 6$ and $n = 7$ respectively; blank areas are zero).

$$\begin{pmatrix} 1 & & & & & -1 \\ & 1 & & & -1 & \\ & & 1 & -1 & & \\ & & 1 & 1 & & \\ & 1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & & & & & & -1 \\ & 1 & & & & -1 & \\ & & 1 & & -1 & & \\ & & & 1 & & & \\ & & 1 & 0 & 1 & & \\ & 1 & 0 & 1 & 0 & 1 & \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

(For odd $n$, inverted V's of 1's and 0's alternate in the lower half.)

Then direct checking shows

$$\mathbf{P}_{2n-1}\mathbf{A}_{2n-1} = \begin{pmatrix} \mathbf{A}_{n-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_n \end{pmatrix} \mathbf{P}_{2n-1}$$

and

$$\mathbf{P}_{2n}\mathbf{A}_{2n} = \begin{pmatrix} \mathbf{D}_n & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_n \end{pmatrix} \mathbf{P}_{2n},$$

establishing similarity. The first and second parts follow immediately. Because $\mathbf{A}_n$ is symmetric and hence diagonalizable (over $\mathbf{R}$) the third part is a consequence of the following lemma.

LEMMA. *If the block diagonal matrix* $\begin{pmatrix} \mathbf{U} & \\ & \mathbf{V} \end{pmatrix}$ *is diagonalizable, then so are* $\mathbf{U}$ *and* $\mathbf{V}$.

This assertion follows from the fact that Jordan forms for $\mathbf{U}$ and $\mathbf{V}$ together yield the *unique* form for $\begin{pmatrix} \mathbf{U} & \\ & \mathbf{V} \end{pmatrix}$, and if the diagonalizability criterion (all Jordan blocks are $1 \times 1$) fails for either $\mathbf{U}$ or $\mathbf{V}$, then it will fail for $\begin{pmatrix} \mathbf{U} & \\ & \mathbf{V} \end{pmatrix}$.

More generally, all the matrices under discussion have distinct eigenvalues (from which diagonalizability follows). This is because the characteristic polynomials of the $\mathbf{A}_n$ form an orthogonal polynomial sequence relative to a positive definite moment functional and hence have simple zeros (see Fevard's Theorem 4.4, Theorem 5.2, and Example 5.7 in *An Introduction to Orthogonal Polynomials* by T. S. Chihara).

Going still further, a complete factorization of these polynomials is available. Subtracting an identity matrix just for convenience, let $a_n$ denote the characteristic polynomial of $\mathbf{A}_n - \mathbf{I}_n$, etc. By the above, $a_{2n-1}(x) = a_{n-1}(x)c_n(x)$ and $a_{2n}(x) = b_n(x)d_n(x)$. Now let $T_n(x) = \cos n\theta$ and $U_n(x) = \sin(n+1)\theta/\sin\theta$ $[x \equiv \cos\theta]$ denote the Chebychev polynomials of the first and second kinds respectively. It turns out that $c_n(x) = 2T_n(x/2)$ and $a_{2n}(x) = U_{2n}(x/2)$. A very readable account of the factorization of $T_n$ and $U_n$ into irreducible polynomials (over $\mathbf{Q}$) is contained in the recently published *Orthogonal Polynomials: From Approximation Theory to Algebra and Number Theory*, *2nd edition*, T. J. Rivlin, John Wiley and Sons, Inc., New York, 1990.

II. *Solution by the proposer.*

Let $T: V \to W$ be a one-to-one linear transformation of finite-dimensional vector spaces. Let $S_V: V \to V$, $S_W: W \to W$ be linear transformations such that $T \circ S_V = S_W \circ T$. Then

(A). $\det(rI - S_V)$ divides $\det(rI - S_W)$,

(B). If $S_W$ is diagonalizable, so is $S_V$.

*Proof of (A).* Let $(r-\lambda)^n$ divide $\det(rI - S_V)$. Then $\dim(\text{nullspace}(\lambda I - S_V)^n) \geqslant n$. We must show $\dim(\text{nullspace}(\lambda I - S_W)^n) \geqslant n$. Suppose $v$ is in the first-mentioned nullspace. Then

$$(\lambda I - S_W)^n(Tv) = T(\lambda I - S_V)^n v = T(0) = 0.$$

Since $T$ is one-to-one, (A) follows.

*Proof of (B).* We need to show that if $(\lambda I - S_V)^n v = 0$ then $(\lambda I - S_V)v = 0$. Suppose $(\lambda I - S_V)^n v = 0$. Then $0 = T(\lambda I - S_V)^n v = (\lambda I - S_W)^n T(v)$. Since $S_W$ is diagonalizable, $(\lambda I - S_W)(T(v)) = 0$. Now $T(\lambda I - S_V)v = 0$. Since $T$ is one-to-one, $(\lambda I - S_V)v = 0$.

To get (i), apply (A) to the following: $V = \mathbf{R}^n$, $W = \mathbf{R}^{2n}$, $S_V = \mathbf{B}_n$, $S_W = \mathbf{A}_{2n}$, and

$$T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ x_n \\ \vdots \\ x_1 \end{pmatrix}.$$

To get (ii) and (iii), apply (B) to $V = \mathbf{R}^n$, $W = \mathbf{R}^{2n-1}$, $S_V = \mathbf{C}_n$, $S_W = \mathbf{A}_{2n-1}$, and

$$T\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \\ x_{n-1} \\ \vdots \\ x_1 \end{pmatrix}$$

*Also solved by Jan Fricke (student, Germany), A. A. Jagers (The Netherlands), Leroy F. Meyers, Ioan Sadoveanu, Heinz-Jürgen Seiffert (Germany), John S. Sumner, William F. Trench, and the Western Maryland College Problems Group.*

*Many solvers noted eigenvalues and roots of Chebychev polynomials.* Meyers *proved the following generalization*: Let $\mathbf{A}_n$ be the $n \times n$ matrix all of whose $(i, j)$ entries are $a$ if $i = j$, and $b$ if $|i - j| = 1$, and 0 otherwise, where $b \neq 0$. Let $\mathbf{B}_n$ and $\mathbf{C}_n$ be like $\mathbf{A}_n$ except that $b_{nn} = a + b$ and $c_{n,n-1} = 2b$. Then the characteristic polynomials of $\mathbf{B}_n$, $\mathbf{C}_n$, and $\mathbf{A}_n$ divide those of $\mathbf{A}_{2n}$, $\mathbf{A}_{2n-1}$ and $\mathbf{A}_{2n+1}$, respectively. Furthermore, each of these matrices is diagonalizable and has distinct eigenvalues. *In addition, it can be shown that the characteristic polynomial of $A_{k(n+1)-1}$ is divisible by that of $A_n$ for every positive integer $k$. These results were the result of conjectures made by experimentation for specific values of n by use of the determinant-free algorithm described by* William A. McWorter, Jr. *in "An Algorithm for the Characteristic Polynomial," this* MAGAZINE, Vol. 56 (1983), 168–175.

# Answers

*Solutions to the Quickies on page 275.*

**A781.** To begin, let us consider the situation in which $P$ is any point in *space*, not in the plane of triangle $ABC$. In this case, the result is the well-known fact that the medians of a tetrahedron are concurrent. The proposed result now follows as the limiting case when $P$ moves into the plane of triangle $ABC$.

**A782.**

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

The right side equals $\lim_{x \to c} f'(x)$ by l'Hôpital's rule.

**A783.** Suppose $n \neq 3$ and $n \neq 6$ but there exists a regular lattice $n$-gon $A_1A_2 \ldots A_n$. Construct equilateral triangles $A_1A_2B_1, A_2A_3B_2, \ldots, A_nA_1B_n$ inside $A_1A_2 \ldots A_n$. It is easy to see that $B_1B_2 \ldots B_n$ is also a regular lattice $n$-gon, and smaller than $A_1A_2 \ldots A_n$. This construction can be repeated to generate a sequence of regular lattice $n$-gons converging to a single point. This is clearly impossible. On the other hand, there certainly exist regular lattice $n$-gons for $n = 3$ and $n = 6$. For $n = 3$, $B_1B_2B_3 = A_1A_2A_3$. For $n = 6$, $B_1 = B_2 = B_3 = B_4 = B_5 = B_6$ and the construction cannot be repeated.

# REVIEWS

PAUL J. CAMPBELL, *editor*
Beloit College

*Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of the mathematics literature. Readers are invited to suggest items for review to the editors.*

**Stewart, Ian, Mathematical recreations: A swift trip over rugged terrain, *Scientific American* (June 1991) 123–125.**

**"*Scientific American* is proud to publish, for the first time, 'Gulliver's Further Adventures on the Flying Island of Laputa.'" Gulliver, it turns out, was clever enough to translate a problem in physics into topology and then use Morse theory to save his neck. Given the delight of this hitherto unknown chapter of *Gulliver's Travels*, I hope that the director of book conservation at the Bodleian Library in Oxford—and indeed, all librarians everywhere—will look hard for further hitherto unknown chapters (and notify columnist Ian Stewart if they find one).**

**Prusinkiewicz, Przemyslaw, and Aristid Lindenmayer, *The Algorithmic Beauty of Plants*, Spring 1990; xii + 228 pp, $39.95. ISBN 0-387-97297-8**

**Magnificent modeling and computer graphics of plants, from layers of cells to trees, from phyllotaxis to lilies on a pond. Rather than the iterated function systems and resulting fractals used by M.F. Barnsley and others, the authors use *L(indenmayer)-systems*, a kind of rewriting or production system. In L-systems, the productions are applied in parallel, rather than sequentially, as in Thue sequences, Chomsky grammars, and most other string-rewriting systems (including J.H. Conway's game of Life). The rewriting system is interpreted in terms of turtle geometry, algorithms accompany the models, and—customary for computer graphics books—beautiful color pictures are included, too. The authors, a theoretical biologist and a computer scientist, have produced a "virtual laboratory in botany."**

**Peterson, Ivars, Computing "fusion trees" to explode barriers, *Science News* 139 (28 June 1991) 406.**

**Michael L. Fredman (UC– San Diego) and Dan E. Willard (SUNY– Albany) have discovered more efficient algorithms for sorting and for searching. Instead of comparing two items at a time, they compare one against many others, by organizing the data into a "fusion tree"; it takes less time to do the organizing than the usual number of comparisons would take. The underlying idea is that a comparison of two numbers usually does not use all the bits of the numbers; so their method selects the relevant bits from a set of numbers and groups the bits into a single expression that can be used in a comparison. Details will appear in the *Journal of Computer and System Sciences*.**

**Arnol'd, V.I., *Huygens and Barrow, Newton and Hooke*, Birkhäuser, 1990; 118 pp, $18.50 (P). ISBN 0-8176-2382-3**

**Examines the context, motivation, and content of Newton's *Principia*. Several astonishing surprises spring from Arnol'd's encyclopedic integrated knowledge of mathematics: 1) l'Hôpital's calculus text "contains a representation of the manifold of irregular orbits of the Coxeter group**

$H_3$ (generated by reflexions in the planes of symmetry of an icosahedron"; 2) the theoretical basis for quasicrystals (which lies in the splitting of the $R^5$ into invariant subspaces induced by a group that fixes a pentagon) completes research begun by Huygens on wave fronts; and 3) Lemma XXVIII of the *Principia* (reproduced on pp. 101–105) contains "an astonishingly modern topological proof of a remarkable theorem on the transcendence of Abelian integrals ... the first 'impossibility proof' in the mathematics of the new era." Arnol'd does not claim that the modern theorems were in the minds of the $17^{th}$-century mathematicians but that fundamental ideas were.

Freedman, David A., Adjusting the 1990 census, *Science* 252 (31 May 1991) 1233–1236. Wolter, Kirk M., Accounting for America's uncounted and miscounted, 253 (5 July 1991) 12–15.

On July 15 the U.S. Secretary of Commerce announced his decision not to adjust the 1990 census, contrary to the recommendations of the Census Director; the adjustment would have shifted a seat each from Wisconsin and Pennsylvania to Arizona and California. Freedman (Statistics Dept., UC—Berkeley) describes the proposed adjustment procedure and concludes "the present state of the art probably cannot support adjustment." Wolter (VP, A.C. Nielsen Co., which does TV ratings), who serves on the special advisory panel to the Secretary, suggests that "corrected counts should be closer than uncorrected counts to the true population." Next: the battle in the courts.

Peterson, Ivars, Searching for new mathematics, *SIAM Review* 33(1) (March 1991) 37–42.

Ivars Peterson is one of the few—and one of the most successful ever—popularizers of mathematics. The author of *The Mathematical Tourist: Snapshots of Modern Mathematics* (Freeman, 1988) and *Islands of Truth: A Mathematical Mystery Cruise* (Freeman, 1990) and a writer for *Science News*, he has just won the Mathematics Communications Award for 1991 awarded by the Joint Policy Board for Mathematics (representing the 55,000 members of the AMS, MAA, and SIAM). In this article he asks: "What are some of the ... factors that stand in the way of understanding and appreciating what mathematicians are up to?" Abstraction and specialized vocabulary, certainly. But the arid format and esoteric style of mathematical papers—far different from papers in other sciences, Peterson points out—"conspire against the broad communication of new mathematical ideas." Also, mathematicians rarely show the human side of their work, or convey the mental imagery and empirical aspect behind it. Asserts Peterson: "Research worth publishing should also be worth communicating."

Selvin, Paul, Does the Harrison case reveal sexism in math?, *Science* 252 (28 June 1991) 1781–1783.

Jenny Harrison, denied tenure in the mathematics department of UC—Berkeley in 1986, sued the university in 1989, alleging that the decision and subsequent appeals procedures reflect a bias against women. The suit takes place against the background fact that only 1% of the tenured faculty—and only 1% of the untenured faculty—in mathematics departments in the "top ten" schools are women.

Cipra, Barry, Math Ph.D.s: bleak picture; Mathematician, heal thyself, *Science* 252 (26 April 1991) 502–503.

Anecdotal evidence suggests that this year there were both a sharply reduced number of jobs for new Ph.D. mathematicians and a glut of applicants . The economic recession and declining number of 18-year-olds account for the reduced demand; while an "unexpected" influx of mathematicians from Eastern Europe and the Soviet Union, plus Chinese students permitted to remain in the U.S., account for the oversupply. As always, there are also the carry-over applicants who took one-year positions the previous year. The result of this confluence of circumstances has been deluge of applications (1,800 at UCLA!). "All this is at odds with forecasts of an impending shortage of mathematicians." Meanwhile, Donald Lewis (U. of Michigan) suggested (*Notices of the American Mathematical Society* 38(4) (April 1991) 296–297) that the mathematical job market should be "rationalised" by agreement on the use of a matching program, like that used to place medical residents; but every known algorithm for such matching favors either the applicants or the institutions.

Cipra, Barry A., Euclidean geometry alive and well in the computer age, *SIAM News* 24(1) (January 1991) 1, 16–17,19.

Relates three recent developments in geometry: solution of the Steiner ratio conjecture (that the minimum length of a Steiner tree is at least $\sqrt{3}/2$ times the length of a minimal spanning tree), a new algorithm for triangulating polygons that is linear in the number of vertices, and a proof that the well-known lattice packing of spheres (with 12 surrounding each one) is the most efficient possible.

Malkevitch, Joseph (ed.), *Geometry's Future*, COMAP, 1991; ix + 105 pp. ISBN none

Goals and recommendations from a conference of leading geometers on what can be done to revitalize college geometry. Their recommendations are mercifully brief (one page), and the bulk of the booklet is devoted to individual essays, which include the outline of one course, the handouts and assignments of a second (by J.H. Conway, P. Doyle, and W. Thurston), and a "dream list" of 60 topics with references for each. If you teach geometry in either high school or college, you need to have this breath of fresh air.

Browne, Malcolm W., In a musical invention, Bach + fractals = new compositions, *New York Times* (16 April 1991) B5, B10. Hsü, Kenneth J., and Andrew Hsü, Self-similarity of the "$1/f$ noise" called music, *Proceedings of the National Academy of Sciences of the U.S.A.* 88 (April 1991) 3507–3509.

"A scientist and his musician have devised a system to condense music to its underlying essence." The system uses—can you guess?—*fractals*. Since fractals, with their self-similarity, involve the repetition of patterns at a change of scale, perhaps one can distill from a fractal the essential pattern, without including all of the repetitions. If this idea can be applied to music, it might tend to vindicate Emperor Franz Josef II, who declared after the initial performance of Mozarts' opera "Abduction from the Seraglio" that there were too many notes. But then again, maybe not .... (Note: The article by Hsü and Hsü bears the footnote, which we should expect to see in mathematics journals, too, that "The publication costs of this article were defrayed in part by page charge payment. This article must therefore be hereby marked '*advertisement*" in accordance with 18 U.S.C. §1734 solely to indicate this fact.")

Gray, L.J., Storage tank design, *SIAM Review* 33(2) 271–274.

Investigates a problem that arose in the design of a storage tank facility for a chemical processing plant. The problem is to arrive at the volume of liquid in a cylindrical tank that is elevated at an angle from the horizontal. The solution uses only integral calculus, aided in the computations by a symbolic computation system. This is a good challenge problem for students in Calculus II!

Tobias, Sheila, *They're Not Dumb, They're Different: Stalking the Second Tier*, Research Corporation (distributed by Science News Books, 1719 N Street N.W., Washington, D.C. 20036), 1990; 94 pp., $2 (P). ISBN none

Reflections, with interpretation, of seven "outsiders" to science who were invited (with pay) to "seriously audit" undergraduate physics and chemistry. They were representatives of the "second tier," bright students who could do science—or mathematics—if they were recruited into it. Author Tobias is concerned with filling the pipeline of scientists. As a result of this study, she recommends active recruitment of the second tier; smaller class sizes; more enjoyable classes, minimal competition, and an established classroom community; larger perspectives, conceptual challenges, and greater intellectual content; and efforts to discourage students from getting off the mathematics train. What also emerges, though, is that students who could do science, but don't, really have different intellectual orientations (trying hard to understand and interrelate concepts, but not wanting to learn the techniques and skill of problem-solving), different learning styles (they actually read the textbook! but are reluctant to memorize "dull" but necessary facts), different motivations (curiosity vs. competition), and different strengths (writing essay examinations and papers vs. the discipline of studying every day).

Huntley, I.D., and D.J.G. James (eds.), *Mathematical Modelling: A Source Book of Case Studies*, Oxford, 1990; xiv + 462 pp, $90. ISBN 0-19-853657-7

A big bonanza of models, for problems as diverse as measuring the length of a roll of toilet paper (a good starter model for a class) to basketball shooting, from why airlines overbook flights to the efficiency of a windmill. The studies, which were prepared by teachers of modeling in workshops do not concisely identify their mathematical prerequisites.

Wagon, Stan, *Mathematica in Action*, Freeman, 1991; xiv + 419 pp, $29.95 (P). ISBN 0-7167-2202-X

Not just long expansions of $\pi$, not just 3-D plots, and not just symbolic algebra: This book shows what the computer program Mathematica can do for problems of interest to mathematicians. You can illustrate the prime number theorem, draw Penrose tiles, examine chaotic behavior of functions, generate fractals, color a torus, obtain and check certificates of primality, determine Eisenstein primes, plot billiard paths on elliptical tables, explore the influence of the zeta function on the distribution of primes, and much, much more. The code for all programs in the book is available from the author for a nominal fee.

Aiton, E.J., *Leibniz: A Biography*, Adam Hilger (distributed in USA by American Institute of Physics), 1985; xiv + 370 pp, $69. ISBN 0-85274-470-6

A thorough biography of Leibniz in English. Treats Leibniz's life in detail, exploring his mathematical contributions but also delving into his other interests, in a rich chronological account.

# NEWS AND LETTERS

LETTERS TO THE EDITOR

Dear Editor:

Roger Nelsen's "Proof without Words: Integration by Parts," which appeared in the April 91 issue, also appears on p. 219 of R. Courant's *Differential and Integral Calculus*, 2nd edition, Interscience, New York, 1937.

Jonathan D. Sondow
Yeshiva University
New York

Dear Editor:

Unfortunately, one of your authors is not up to date. Houston Euler, the eminent Texas mathematician, wrote on "The History of 2 + 2 = 5" in your December 1990 issue. It is not surprising that he is unaware of recent work done on this equation at Princeton. I did not do the work, I only reported the outcome of work done by others.

Like Martin Luther, the author of this work posted his thesis in a prominent public place, but perhaps because of public controversy, the author chose to remain anonymous. I stumbled onto this quite by accident and until I published my account of it, this work has been studied only by those who happened to have encountered the thesis as it was originally posted.

With modest study, you should be able to see that the unknown author has made a great contribution to the understanding of this equation. He has put it on a sound theoretical ground so that from this foundation it should be possible to begin to explore the full ramifications of this breakthrough.

Albert A. Bartlett
University of Colorado
Boulder, CO

*Enclosed with his letter, Professor Bartlett included p.634 from* The Physics Teacher, *November 1989, which ran this blurb:*

*"Jim Faller tells of an inscription on the wall of a restroom at Princeton:*

*2 + 2 = 5 for large values of 2."*

*–Ed.*

Dear Editor:

While doing a problem, I stumbled upon this approach to Kummer's Theorem concerning the divisibility of multinomial coefficients by a fixed prime. This seems a bit more straightforward than the method of [1]. As is done in that article, we denote the highest power of a prime $p$ that divides an integer $k$ as $\mathrm{ord}_p(k)$.

**Theorem.** The number of carries in the $p$-ary addition of $e_1, e_2, \ldots, e_t$ equals

$$ord_p\binom{n}{e_1 e_2 \ldots e_t}.$$

**Proof.** We again start with the well-known result that the highest power of a prime $p$ that divides $n!$ is $\sum_{j=1}^{\infty}\lfloor\frac{n}{p^j}\rfloor$. Let $c_i$ be the carry in the $i$-th column of the addition of $e_1, e_2, \ldots, e_t$ in base $p$. It is easy to verify that

$$c_i=\lfloor\frac{n}{p^{i+1}}\rfloor-\lfloor\frac{e_1}{p^{i+1}}\rfloor-\lfloor\frac{e_2}{p^{i+1}}\rfloor-\ldots-\lfloor\frac{e_t}{p^{i+1}}\rfloor.$$

If now $n=e_1+e_2+\ldots+e_t$, then

$$ord_p\binom{n}{e_1 e_2 \ldots e_t}=$$

$$ord_p(n!)-ord_p(e_1!)-\ldots-ord_p(e_t!)$$

$$=\sum_{j=1}^{\infty}\lfloor\frac{n}{p^j}\rfloor-\lfloor\frac{e_1}{p^j}\rfloor-\ldots-\lfloor\frac{e_t}{p^j}\rfloor=\sum_{i=0}^{\infty}c_i,$$

as desired.

**Reference:**

1. F. Dodd and R. Peele, Some counting problems involving the multinomial expansion, this MAGAZINE 64 (1991), 115-122.

Kiran S. Kedlaya
Silver Spring, MD

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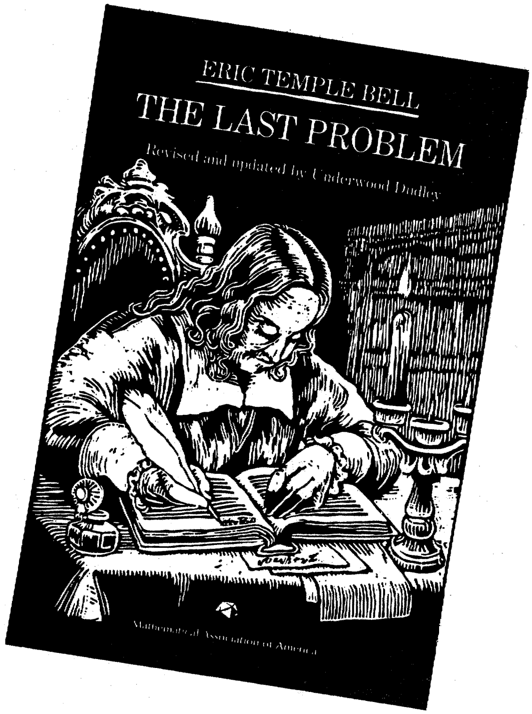


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